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The Materials in this bulletin indicate suggested teaching procedures needed to implement the teaching of "mathematics, 9th Year" as outlined in Curriculum Bulletin No. 3, 1958-59 series, Course of Study Mathematics 7-8-9. Whereas the course of study suggests the application of mathematical principles such as commutativity, associativity, and distributivity to algebraic skills and techniques, in this bulletin detailed methods for helping pupils to develop these mathematical concepts are given. Topics include symbols, signed numbers, algebra, polynomials, equations and inequalities, equations and graphs, factoring, fractions, real numbers, quadratic equations, ratio and proportion, and indirect measurement. (RP)

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Mathematics

9th YEAR

Board of Education · City of New York

SE005 410

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FOREWORD

At a time when our society is increasingly more dependent upon mathematically literate citizens and upon trained mathematical manpower, it is essential that vital and contemporary mathematics be taught in our schools.

The contemporary mathematics program set forth in this publication has developed as a result of experimentation and teacher and supervisor evaluation in classroom situations.

The keynote of the program is understanding of structure, language, skills, and basic mathematical concepts. This understanding is developed in an atmosphere of pupil experimentation and discovery.

We wish to thank the staff members from the Junior and Senior High Schools and the Bureau of Curriculum Development who have so generously contributed to this work.

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INTRODUCTION

This bulletin is the culmination of three years of experimentation involving the cooperative efforts of the Division of Curriculum Development, the Junior High School Division, and the High School Division. It offers teachers and supervisors practical suggestions for teaching Mathematics Ninth Year. The mathematics presented in this bulletin is based upon concepts and skills which were developed in previous grades and will be extended in succeeding grades. It is one segment of a K-12 mathematics program.

The suggested teaching procedures help to implement the teaching of Mathematics Ninth Year as outlined in Curriculum Bulletin No. 3, 1958-59 Series, Course of Study Mathematics 7-8-9. Detailed methods for helping pupils develop mathematical concepts are given, to an extent not spelled out in the course of study. Thus, whereas the course of study merely suggests the application of mathematical principles such as commutativity, associativity, and distributivity to algebraic skills and techniques, these principles are carefully developed and woven into these materials.

The course is a full course and teachers and supervisors must consider this as they plan for pupil participation in the discovery and creation of ideas which are then organized into a growing structure. To make the course manageable by classroom teachers, such topics as congruence, symmetry and statistics were omitted. A treatment of inequalities has been included.

The materials in this publication have been developed over a period of years and reflect the classroom tryout and continued evaluation by teachers and supervisors. There is an emphasis on:

an understanding of mathematical principles
a development of manipulative skills based upon mathematical principles
mathematical structure
growth of number system
precise meaning of vocabulary
justification, or proof

The level of mathematical maturity of the pupils has been considered and the approach and amount of rigor introduced is consistent with the capacity of ninth year pupils. The 10th and 11th year mathematics courses will extend the basic ideas of this course and help them to see algebra as a postulational system.

How This Bulletin Is Organized

The material in this bulletin is arranged in the same sequence in which it is to be used by the teacher. It is expected that the entire content of each chapter will be presented before any work is begun on the ensuing chapter. (The few exceptions to this procedure are noted in the appropriate place.) Although this may seem to be a departure from the cyclical arrangement of materials found in earlier curriculum bulletins on mathematics, a cyclical approach is, in fact, an integral part of each chapter. For example, understanding and skill in solving equations are developed on progressively higher levels throughout the year as pupils advance from the solution of simple linear equations, presented in earlier chapters, to the solution of fractional and quadratic equations which appear in later chapters.

Various suggestions for enrichment have been included. Labeled as optional, they have been placed near the topics of which they are a logical outgrowth.

A Plan For Using This Publication

It is suggested that teachers and supervisors consider the following in using this publication:

Review the entire bulletin before making plans to teach. Read each chapter in turn completely to become acquainted with the content and flavor of Mathematics Ninth Year and with the relationship between the topics in the course.

Study the chapter containing the topics you plan to present. Study the suggested procedure for the development of each topic. Make a tentative division of the topic into class lessons. Use the suggested procedures as an aid in preparing lesson plans. Since Mathematics Ninth Year is a full course, you should plan for the most efficient use of class time.

Amplify the practice material suggested for each topic with additional material from various textbooks.

The application of algebraic techniques to the solution of verbal problems should not be confined to the sections in which this work appears in the bulletin, but should be interspersed among other topics in order to sustain interest and provide for continuous development and reinforcement of problem-solving skills.

EVALUATION

An evaluation program includes not only the checking of completed work at convenient intervals, but also continual appraisal. It is a general principle of evaluation that results are checked against objectives. The objectives of this course include concepts, principles, and understandings, as well as basic skills.

Written tests are the most used instrument for evaluation and remain the chief rating tools of the teacher. Test items should be designed to test not only recall of factual items, but also the ability of the pupil to make intelligent use of facts. Some of the written activities which teachers may use for the purpose of evaluation include:

written tests
written homework assignments
notebooks
board work
special reports
quizzes

To continually evaluate pupil understanding, there are a number of oral activities which teachers may use including the following:

pupil explanations of approaches used in new situations
pupil justification of statements
pupil restatement of problems
pupil statements of interrelationship of ideas
pupil discovery of patterns
oral quizzes
reports

Evaluation procedures also include teacher's observation of pupil's work at chalkboard and pupil's work at seat.

Self-evaluation by pupils can be encouraged through short self-marking quizzes.

DEVELOPMENT OF THE MATHEMATICS PROGRAM IN GRADE 9

During the school year beginning September 1961, a revised ninth year mathematics scope and sequence, based upon Course of Study Mathematics 7-8-9, was developed by staff members from the Division of Curriculum Development, the Junior High School Division, and the High School Division. This scope and sequence was the basic document for writing teams, which consisted of junior high school mathematics coordinators working in conjunction with high school mathematics chairmen. This resulted in preparation of preliminary materials which were reviewed and revised by the Junior High School Mathematics Curriculum Committee. By June 1962, the first draft of materials was ready and was sent to teachers who were to take part in their experimental use.

These preliminary materials were tried out on an experimental basis for the first time in selected junior and senior high schools during the school year 1962-1963. A program of evaluation of these materials was set up which included: chapter by chapter evaluation reports from classroom teachers, junior high school coordinators, and chairmen of mathematics in pilot schools. The results of the evaluation were reported to the Junior High School Mathematics Curriculum Committee. In addition, an evaluation team interviewed teachers using the materials, observed teachers using the materials, and discussed each observed lesson with the teacher. On the basis of this classroom tryout and evaluation by teachers and supervisors, Part I of the program was revised in the Summer of 1963.

The school year 1963-1964 saw the second year of experimental use of the materials, with additional schools participating. Part II of the material was revised in time for the February 1964 term. Part I was again revised as a result of a second tryout. This revision was carried out by a committee working on Saturdays. Final work on Part I, preparing it for publication, was completed in June 1964.

It is expected that a final revision of Part II will be completed during the Summer of 1964. This revision will reflect the second year of classroom tryout.

The preparation of this bulletin was under the general direction of Martha R. Finkler, Acting Associate Superintendent, Junior High School Division, and Margaret Bible, Acting Assistant Superintendent, Junior High School Division, William H. Bristow, Assistant Superintendent, Bureau of Curriculum Research, and Seelig Lester, Assistant Superintendent, High School Division.

As Chairman of the Junior High School Mathematics Curriculum Committee, Paul Gastwirth, Principal of Edward Bleeker Junior High School, acted as project director, coordinating efforts of various committees, and heading a four-man evaluation team.

Frank J. Wohlfort, Assistant Principal in charge of Junior High School Mathematics Coordinators, coordinated the work of the writing teams, and arranged for experimental tryout of the program in the junior high schools.

Miriam Newman, Junior High School Mathematics Coordinator, was the principal writer of the revised materials during the Summer of 1963, the school year 1963-1964, and the Summer of 1964.

The Junior High School Mathematics Coordinators who prepared the original draft of the materials were: Spencer J. Abbott, Henrietta D. Antoville, Florence Apperman, Samuel Bier, Samuel Dreskin, Charles S. Goode, Helen Halliday, Ida Karlin, Rose Klein, Miriam S. Newman, Alfred Okin, George Paley, Meyer Rosenspan, Benedict Rubino, Joseph Segal, and Murray Soffer.

Other coordinators who have taught and helped in the evaluation of the materials are: Helen Kaufman, Joseph Gehringer, Ada Sheridan, and Bertha Weiss.

The original writing teams worked closely with the following high school mathematics chairmen: George Grossman, Aaron Hankin, Roxee Joly, Harry Ruderman, Lester Schlumpf, and Harry Schor. Each of the chairmen served as a resource person for a writing team.

George Ross, Coordinator of Mathematics for the high schools during the school year 1961-1962, was a member of the original planning group which developed the plans for the project and the basic scope and sequence. An initial scope and sequence was prepared by Harry Schor, Chairman of Mathematics.

Benjamin Bold, Coordinator of Mathematics for the high schools during the school years 1962-1964, became a member of the planning group and coordinated the high school efforts in the project. He took part in the summer writing projects of 1963 and 1964.

Leonard Simon, Junior High School Curriculum Coordinator, Bureau of Curriculum Research, was a member of the original planning group and has continued through the program to assist in the planning, coordinating, revising, and preparing materials for publication.

Grateful acknowledgment is expressed to the many who have contributed to this bulletin: the present and past members of the Junior High School Mathematics Curriculum Committee, High School Standing Committee on Mathematics, principals of experimental schools who arranged schedules so that their school could take part in the experimentation, and made it possible for the evaluation team to meet with and observe teachers.

Teachers and supervisors who used the material in classroom tryout and who played a part in the evaluation and revision include:

Sydni Abramowitz, Salvatore Attanasio, Emily Bipple, Alice Blume, Lula Bramwell, Julian Bronstein, Helen Broudy, Ellen Bruckner, Arthur Bryton, Ralph Caporale, Isadore Chenat, Anna Chuckrow, Saris Cohen, Richard Colasuonno, Arthur Crean, Helen Davis, Matilda Delise, William Doherty, Beatrice Duberstein, Minnie Feldman, Sally Finkelstein, Bennett Fisch, June Fleary, Rochelle Ganz, Sidney Gelman, Lydia Gersten, Phyllis Glanz, Leo Glickman, Milton Gluck, Mildred Goldberg, Martin Goffe, Max Greenspan, Sylvia Greenwald, Jennie Gross, Estelle Gurin, Gerard Haggerty, Doreen Hall, Dennis Hayes, Thelma Hickerson, Abe Hoberman, Ella Huerstel, Estelle Hurwith, Robert Jacobson, Leale Joffe, Lillian Joseph, Elizabeth Kelly, Herman Kirshbaum, Irving Klein, Meyer Klein, Manfrid Korman, Ruth Kranz, Herbert Krasnoff, Peter Krauss, Dorothy Kreiger, Mildred Lawton, Frieda Lazarus, Louis Leen, Adeline Lesk, Herbert Linchitz, Myra Mandel, Bernard Mannes, Rosaline Merzer, Blanche Michnick, Elaine Mintz, William Morrill, Joyce Murray, Irving Myron, Ruth Newfield, Iris Ort, Max Paskin, Robert Perkus, Alan Plank, Anthony Polomena, Stephen Raucher, Alfred Reinfelder, Etta Reynolds, Selma Rhine, Bertha Rhodes, Beverly Ribaro, Lillian Robinson, Jerome Rosavsky, Jules Rowen, Richard Rubin, Jay Sachs, Myra Sachs, Bernard Schrenzel, Doris Schwartz, Marion Scully, Monty Seewald, Alice Shuger, Hannah Sidofsky, Irwin Simon, Alfred Smith, Shulamith Stechel, Bernard Storch, Felice Thurman, Max Tufel, Peter Tymus, Mildred Vogt, Rose Wantman, Joan Warshaw, Rose Werer, Ruth White, Arlene Young, Harry Zekster.

A special acknowledgment is made to Frances Moskowitz, who prepared for production the original, the revision, and the final form of these materials.

Simon Shulman designed the cover.

Maurice Basseches, Editor, had over-all responsibility for design and production.

CHAPTER I

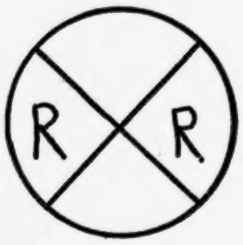
SYMBOLS IN MATHEMATICS

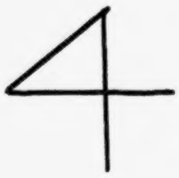
This section contains materials and suggestions for the teacher to help the pupils understand the meaning of numbers and numerals, symbols of grouping, and conventions for omitting grouping symbols. These concepts are then used in forming mathematical sentences.

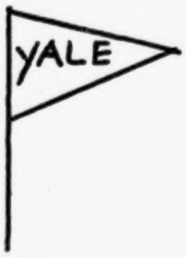
I. Numbers and Numerals

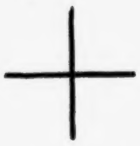
A. Suggested Procedure

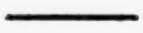
1. Have pupils examine the following symbols and tell what they represent:











Abraham Lincoln   honesty

2. Have pupils realize that a symbol such as a mark, a sign, or a word may represent an object, a person, or an idea.

3. Have pupils write in their notebooks, and on the blackboard, several symbols for real objects or persons, and several symbols for ideas. Have pupils explain what each symbol represents.

4. Through group discussion, develop the understanding that several different symbols may represent the same object, person, or idea.

   For example, "Abraham Lincoln" and "Honest Abe" are different names or symbols for the same person.

5. Elicit the generalization that the symbol is distinct from the object, person, or idea represented. For example, when a pupil writes "Abraham Lincoln" on his paper, the name of a person is on the paper, not the person himself.

6. Similarly, have pupils develop an understanding of the difference between a *number* (idea) and its *name*. Some of the many possible number names for a set of five objects are "5", "five", "V", and "2+3".

7. Help pupils to understand that just as "Abraham Lincoln" or "Honest Abe" are symbols or names used to designate the same person, so "five", "5", and "V" are different names for the same number.

8. Lead pupils to realize that when they write symbols as "7", "6+4", and "55", they are writing *names* or *symbols* for numbers, not the numbers themselves. Numbers are *ideas* for which the symbols stand.

9. Explain that the names or symbols for numbers are called *numerals*, and that these numerals may be spoken or written.

10. Make sure that pupils understand that each number has many different names. Thus, such numerals as "5+1", "3+3", "7-1", "$\frac{12}{2}$", "2x3", and "VI" all *name* the *same* number, namely, six.

11. Have pupils indicate the common names for some numbers; have other pupils give different names for those same numbers.

B. Suggested Practice

1. Orally, or in written form, suggest a symbol for each of five different objects.

2. Orally, or in written form, suggest a symbol for each of five different ideas.

3. Orally, or in written form, suggest a symbol for each of five different people.

4. Think of a number. Write three different numerals for this number.

5. On each line place a circle around the numeral that does not belong because it names a different number.

a. 3 + 1	2^2	10 - 6	10 ÷ 5	$\frac{8}{2}$
b. 6 x 1	6 + 0	$2\frac{1}{3} + 3\frac{2}{3}$	12 - 7	600%
c. .05	$\frac{1}{10} + \frac{1}{2}$	$\frac{1}{20}$	$\frac{10}{200}$	5%

d. Make up two examples like a, b, and c above.

6. Write another name for each of the following:

a. $\frac{1}{2} + \frac{1}{4}$

b. $.37 - .32$

c. $\frac{5}{8} \times \frac{1}{2}$

d. $10 - .87$

e. 10^2

f. $14 \div 3\frac{1}{2}$

g. $.85 \times 1.5$

h. $.35\overline{)245}$

i. $\frac{1}{2} - \frac{1}{5}$

j. $1.45 + .07$

II. Symbols Used in Arithmetic

A. Suggested Procedure

1. Review with pupils the symbols of operation and their meaning:

$+ \qquad - \qquad \times \qquad \div$

a. Elicit from pupils an explanation of each of the symbols of operation as follows:

$2 + 3$ is the number obtained when 3 is added to 2

6×5 is the number obtained when 5 is multiplied by 6

b. Elicit from pupils the generalization that each of these operations involves two numbers (binary operation).

2. Review with pupils the following symbols of comparison and their meaning:

$= \qquad \neq \qquad > \qquad <$

a. Guide pupils to see that when they write the sentence $3 + 4 = 7$, they are saying that "3 + 4" and "7" are names for the same number. The symbol "=" means that the numeral on its left and the numeral on its right are both names for the same number. The symbol "=" is read as "is equal to."

b. The symbol "≠" means "is not equal to." For example, $3 + 4 \neq 6 + 2$ means that the numeral "3 + 4" and the numeral "6 + 2" represent different numbers.

c. The symbol "$>$" means "is greater than." For example, $5>4$ means that the number represented by the numeral "5" is greater than the number represented by the numeral "4."

d. The symbol "$<$" means "is less than." For example, $4<7$ means that the number represented by the numeral "4" is less than the number represented by the numeral "7."

B. Suggested Practice

1. Mark each statement with a T or F to indicate whether it is true or false.

   a. $10^2 = 100$ T ("10^2" and "100" name the same number.)

   b. $3 + 7 \neq 2 + 6$

   c. $1.06 > 10.6$

   d. $5 \times 4 \neq \frac{1}{5}$ of 100

   e. $\frac{1}{8} + \frac{1}{4} = .375$

   f. $2.00 - .50 < 1\frac{1}{4}$

   g. $1 \times 1 > 1 + 1$

   h. $4^2 = 15 + 1$

   i. $10^3 > 1000$

   j. $\frac{1}{10}$ of $20 < 2^2$

2. Fill in each space with any of these symbols ($+$, $-$, $\times$, $\div$) which will make the resulting statements true.

   a. 3 ____ $5 < 14$

   Solution: If the symbol "+" is placed between 3 and 5, the statement will read $3 + 5 < 14$. This is a true statement. Also, $3 \div 5 < 14$ is a true statement.

   b. 4 ____ $3 = 1$

   c. 10 ____ $5 < 3$

   d. $\frac{3}{4} < 1$ ____ $\frac{1}{2}$

   e. 8 ____ $7 > 8 - 7$

   f. 20 ____ $4 < 20 - 4$

   g. 4 ____ $4 = 1$

   h. 2 ____ $2 = 2^2$

   i. 6 ____ $0 = 6$

III. Order of Operations

A. Suggested Procedure

1. Pose problem: What number is represented by the expression $3 + 6 \times 5$?

   Some pupils may suggest 45 because $3 + 6 = 9$, and $9 \times 5 = 45$. Others may suggest 33 because $6 \times 5 = 30$, and $3 + 30 = 33$.

   Have pupils see that the difference in the two answers is due to the order in which the operations are performed.

2. Elicit that if we are to get the same meaning from $3 + 6 \times 5$ at all times, we must agree on the order in which operations are performed. Inform pupils that when several operations are indicated in an expression, we proceed as follows:

   a. Perform the multiplications and divisions in order, from left to right.

   b. Then, perform the additions and subtractions in any order.

   Using this agreement, $3 + 6 \times 5$ always represents 33.

3. Have pupils practice the following:

   What is the meaning of each of these expressions? What is the result?

   a. $1 + 4 \times 5$ (This means the product of 4 and 5 is to be added to 1. The result is 21.)

   b. $9 \times 8 - 7$

   c. $45 \div 3 + 2$

   d. $8 \times 4 + \frac{1}{2}$

   e. $-6 + 9 \div 3$

4. Have pupils see the need for a grouping symbol when the agreement on order of operations is to be disregarded.

   a. Pose problem:

      On each of four Saturdays, a boy earned \$8 and collected \$2 in tips, as well. What is the total amount he received? (\$40) How would you indicate in symbols how the answer is arrived at?

      Pupils may suggest $4 \times 8 + 2$. Have them recall that by the agreement on order of operations, this expression results in 34, not 40.

   b. Guide pupils to realize that if we wish $4 \times 8 + 2$ to mean that the sum of 8 and 2 is to be multiplied by 4, giving the result 40, rather than 34, then we must indicate in some way that the sum of 8 and 2 is to be considered as a single quantity to be multiplied by 4. To show this meaning, $8 + 2$ is enclosed within grouping symbols, such as parentheses. Then $4 \times (8 + 2)$ means that 2 is to be added to 8, and the resulting number is to be multiplied by 4.

      $$4 \times (8 + 2) = 4 \times 10 = 40$$

   c. Have pupils understand that one use of grouping symbols is to give an expression a meaning other than the one it would have according to the agreement on order of operations.

5. Have pupils answer the following:

   a. What is the meaning of $4 + 3 \times 2$? (This means the product of 3 and 2 is to be added to 4. What is the result? (10) How should this be written to give 14 as a result? $(4 + 3) \times 2$.

   b. What is the meaning of $9 \times 8 - 5$? What is the result? (67) How should this be written to give 27 as a result? $9 \times (8 - 5)$.

   c. What is the result for $2 + 6 \div 2 \times 1$? (5) How should this be written to give 4 as a result? $(2 + 6) \div 2 \times 1$.

   d. What is the result for $6 - (2 - 1)$? How should this be written to give 3 as a result?

   e. What is the difference in meaning of each expression in the following pairs? What is the result in each?

$25 - (9 - 3)$	$25 - 9 - 3$
$10 \times (2 + \frac{1}{2})$	$10 \times 2 + \frac{1}{2}$
$4 + (5 - 1)$	$4 + 5 - 1$
$(4 + 8) \div 2$	$4 + 8 \div 2$
$.3 \times (10 + 1)$	$.3 \times 10 + 1$

5. Have pupils compare the order in which the operations in the following expressions are to be performed. Have them then do the indicated arithmetic.

a. $5 + 1 \times 2$ (Multiply, then add)	$(5 + 1) \times 2$ (Add, then multiply)
b. $3 \times 4 - 2$	$3 \times (4 - 2)$
c. $(20 - 8) \div 2$	$20 - 8 \div 2$
d. $(25 - 20) \div 5 + 3$	$25 - 20 \div 5 + 3$
e. $12 \div 6 \times 3$	$12 \div (6 \times 3)$

B. Suggested Practice

1. Write the most common name for each of the following:

   a. $1 + 3 \times 7 = 1 + 21 = 22$
   b. $42 - 2 \times 7$
   c. $8 \times 6 - 7 \times 4$
   d. $2 \times 2 + 2 \div 2$
   e. $1 \times \frac{1}{4} + \frac{1}{4}$
   f. $.5 \times 2 + .3 - .3$

2. For each of the following, replace the expression in the parentheses by another numeral.

   a. $(6 + 2) \times 8 = 8 \times 8$

   b. $(6 + 2) + 4$

   c. $15 - (6 + 4)$

   d. $6 + (2 - 1)$

   e. $5 \times (4 + \frac{1}{2})$

   f. $125 - (5 - 4)$

   g. $12 \div (4 - 2)$

   h. $10 \times (1 + .2)$

   i. $(11 \times 4) \div (2 \times 2)$

   j. $(12 - 3) \times (6 + 3)$

3. Use parentheses in expressing each of the following in symbols:

   a. Indicate that 3 and 8 are to be added and then 10 is to be subtracted from this sum.

   b. Indicate that 4 and 9 are to be added and this sum is to be multiplied by 3.

   c. Indicate that 15 is to be subtracted from the product of 7 and 3.

   d. Indicate that the sum of 6 and 9 is to be multiplied by the difference of 11 and 4.

   e. Indicate that the product of 6 and 7 is to be subtracted from the product of 8 and 9.

4. In each of the following examples perform the indicated operations.

   a. $6 - 2 + 1$

   b. $5 + 3 \times 3$

   c. $10 \times (1 + 4)$

   d. $4 - 3 \times \frac{1}{2}$

   e. $(15 - 6) \div 3$

   f. $15 - 6 \div 3$

   g. $(\frac{1}{2} + 2) \times 4 - 8 \div 4$

   h. $(.4 + 1.1) \div (.5 - .2)$

   i. $10 - 1.5 \times 6 + (14 - 4) \div 2$

   j. $(12 - 3 + 6 \div 2) \div 4 \times 3$

IV. Translation of English Phrases and Sentences

A. Suggested Procedure

1. Discuss with pupils the advantage of using mathematical symbolism to express number relationships. For example, the number relationship.

   "When three is added to nine, the result is twelve" may be expressed symbolically as $9 + 3 = 12$.

2. Have pupils use the conventions relating to grouping symbols to express number ideas symbolically, e.g.,

"When four is added to six times two, the result is sixteen" is expressed symbolically as $6 \times 2 + 4 = 16$.

"The sum of eight and four, divided by three is less than five" is expressed symbolically as $(8 + 4) \div 3 < 5$.

B. Suggested Practice

1. Express each of the following in mathematical symbols:

   a. When three is added to nine times seven, the result is sixty-six.

      Answer: $9 \times 7 + 3 = 66$

   b. Three tenths diminished by one tenth is less than five tenths.

   c. Twice fifteen, decreased by eight, is less than three times ten.

   d. The product of five hundredths and six is not equal to three.

   e. When ten is divided by two and the result is divided by one-half, the quotient is ten.

   f. The difference between seven and two, divided by five, equals one.

   g. When one-half is added to one-fourth multiplied by three, the result is equal to one and one-quarter.

   h. The product of one and one-quarter, increased by one-quarter, is greater than three- eighths.

   i. When the quotient of three-tenths and three-tenths is added to the product of two and five-tenths, the sum is equal to two.

   j. The difference of five and two, divided by the quotient of six and two, is equal to one.

2. Express each of the following in English phrases and sentences:

   a. $8 + 2 \times 3$ Answer: Eight increased by the product of two and three, or the product of two and three added to eight.

   b. $5 + 9 < \frac{1}{3} \times 60$

   c. $(12 - 4) \times 2$

   d. $9 \div \frac{3}{4} > 6$

   e. $4 \times 1 = 16 \div 4$

CHAPTER II

SIGNED NUMBERS

This section contains suggestions for procedures that the teacher may use to help the pupils understand operations with signed numbers and some basic properties of signed numbers.

I. The Set of Numbers of Arithmetic Enlarged to Include the Negative Numbers

A. Suggested Procedure

Review with pupils the concepts of signed numbers that were taught in Grade 8. Emphasize the symbols for and physical interpretation of these numbers.

1. An understanding of the meaning of signed numbers
2. An understanding of the one-to-one correspondence between the integers and certain points on the number line
3. An understanding of the concept of "opposites." Thus, -7 (read "negative seven") is the opposite of $+7$ (read "positive seven"), and $+7$ is the opposite of -7.
4. An understanding of the dual interpretation given to "+" and to "-", e.g., as signs of operation and as signs indicating direction
5. An understanding of order: $-1 > -2$.

B. Suggested Practice (Review Exercises)

1. If $-10{,}000$ represents a decrease of 10,000 in population, represent by a signed number an increase of 10,000 in population.
2. If $+5$ represents an increase of 5° in temperature, represent by a signed number a drop of 5° in temperature.
3. If $+5$ represents a gain of 5 hours in time, represent by a signed number a loss of 5 hours.
4. If -10 represents a withdrawal of \$10 from a bank, represent by a signed number a \$10 deposit in the bank.
5. If $+500$ represents 500 feet above sea level, represent by a signed number 500 feet below sea level.
6. If $+10$ represents a gain of 10 yards, represent by a signed number a loss of 10 yards in a football game.

7. If -27 represents 27° below zero, represent by a signed number 27° above zero.

8. If -10 represents a decrease of 10 cents in the price of eggs, represent a 10 cent increase in the price of eggs.

9. If traveling north is traveling in a positive direction, express as a signed number the location of the point reached after traveling 5 miles north from zero and then 8 miles south.

10. Is $-4 > 0$ a true sentence?

II. Addition Is Commutative for the Set of Signed Numbers

A. Suggested Procedure

Note: In their computations in arithmetic, pupils have used the commutative and associative properties of addition and multiplication, perhaps without recognizing these properties or knowing their names. Now they will learn not only the names of these basic properties, but also that these properties continue to hold in our enlarged set of numbers. These properties will be accepted without proof.

1. Have pupils review addition of signed numbers and the meaning of absolute value. (See Mathematics Grade 8, Curriculum Bulletin 1961-62 Series No. 4)

2. Discuss with pupils a method of checking an arithmetic addition example, by reversing the addends. Thus, to check

$$\begin{array}{r} 389 \\ +\underline{125} \\ 514 \end{array}$$

we could do the addition in this way:

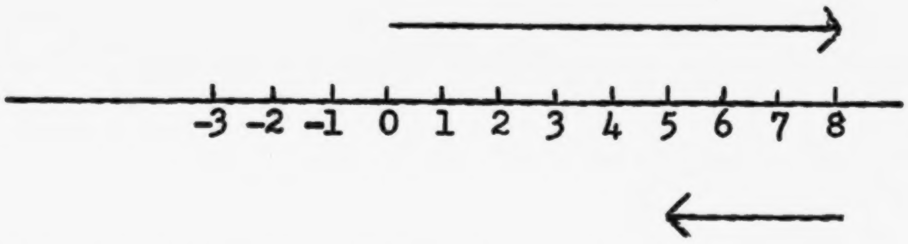
$$\begin{array}{r} 125 \\ +\underline{389} \\ 514 \end{array}$$

(Note: Many pupils check arithmetic addition by "adding in the opposite direction." This is tantamount to reversing the addends.)

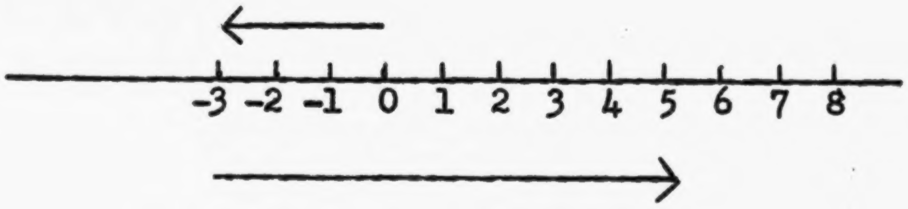
Guide pupils to see that in arithmetic addition, they have assumed that when two numbers are added, the same sum is obtained no matter what order is used in adding them. This is known as the commutative property of addition. Does this property hold for signed numbers, as well?

3. Illustrate the commutative property of addition with positive and negative numbers as follows:

Mr. B walked 8 blocks to the east and then turned around and walked three blocks to the west. If he had first walked three blocks to the west and then 8 blocks to the east, would he have arrived at the same place?



8 blocks east and then 3 blocks west may be represented by $(+8) + (-3)$. The sum is $+5$.



3 blocks west and then 8 blocks east may be represented by $(-3) + (+8)$. The sum is $+5$.

The pupils are led to observe that interchanging the order of the two addends does not affect the sum. Then $(+8) + (-3) = (-3) + (+8)$.

On the basis of several such illustrations, have pupils generalize that the commutative property of addition holds for positive and negative numbers in all combinations. (Or, briefly, for signed numbers)

Inform pupils that the commutative property of addition is sometimes referred to as the commutative law or the commutative principle of addition.

B. Suggested Practice

1. Use either the symbol $(+7)$ or the symbol $(+5)$ to replace the question mark so that each of the following is a true statement.

$(+7) + (+5) = (+5) + ?$

$(+7) + (+5) = \ ? \ + (+7)$

$(+7) + \ ? \ = (+5) + (+7)$

$? \ + (+5) = (+5) + (+7)$

2. Use the commutative principle to write another expression for each sum, then compute the sum.

   $(+7) + (+5) = (+5) + (+7) = +12$

   $(-18) + (-24)$

   $(+23) + (-9)$

   $(-1.2) + (4.8)$

   $(+2\frac{1}{2}) + (-1\frac{3}{4})$

III. Addition is Associative for the Set of Signed Numbers

A. Suggested Procedure

1. Review with pupils the fact that the fundamental operations of arithmetic, i.e., addition, subtraction, multiplication, division are binary operations, that is, they are performed on two numbers. (In contrast, taking the negative of a number is a unary operation.)

2. Have pupils give illustrations of the fact that we often have occasion to add three numbers.

   Pose the question: How can we add three numbers when addition is a binary operation?

3. Elicit from pupils that when we add 5, 9, and 6, we are really performing the operation of addition twice, as follows:

   $5 + 9 = 14$ and $14 + 6 = 20$

4. Have pupils compute an addition example such as the following:

   $49 + 27 + 3$

   In checking the way various pupils arrived at the answer, it will be found that some pupils may have added 27 to 49, obtaining 76. They then added 3 to this result. This procedure may be recorded thus:

   $(49 + 27) + 3 = 76 + 3 = 79$

   Other pupils, using decade facts, may have added 3 to 27, obtaining 30. They then added 30 to 49. We record this as:

   $49 + (27 + 3) = 49 + 30 = 79$

5. Have pupils observe that $(49 + 27) + 3 = 49 + (27 + 3)$, since the sum is the same in each case.

6. Have pupils repeat this procedure with other examples. They see that the manner in which the numbers of arithmetic are grouped in addition does not affect the sum. Notice that the order of the addends is the same.

7. Inform pupils that this property of addition for arithmetic numbers is called the associative property. The word "associative" is used because this property is concerned with how numbers are associated or grouped together. Grouping symbols (parentheses) may be omitted in indicating the sum of three numbers.

8. Now pose this question: Does the associative property of addition hold for signed numbers? Guide pupils to an answer by having them do the following examples:

   Note to teacher: Explain to pupils that the parentheses are used to make clear the sign of the number. The symbol of grouping in these examples ia a pair of brackets.

   a. Add: +3, +2, +5

      $(+3) + (+2) + (+5)$ means $\left[(+3) + (+2)\right] + (+5)$.

      Does $\left[(+3) + (+2)\right] + (+5) = (+3) + \left[(+2) + (+5)\right]$?

      Since $\left[(+3) + (+2)\right] + (+5) = (+5) + (+5) = 10$

      and $(+3) + \left[(+2) + (+5)\right] = (+3) + (+7) = 10$

      then $\left[(+3) + (+2)\right] + (+5) = (+3) + \left[(+2) + (+5)\right]$

   b. Add: -6, -3, -4

      $(-6) + (-3) + (-4)$ means $\left[(-6) + (-3)\right] + (-4)$.

      Does $\left[(-6) + (-3)\right] + (-4) = (-6) + \left[(-3) + (-4)\right]$?

      Since $\left[(-6) + (-3)\right] + (-4) = (-9) + (-4) = -13$,

      and $(-6) + \left[(-3) + (-4)\right] = (-6) + (-7) = -13$,

      then $\left[(-6) + (-3)\right] + (-4) = (-6) + \left[(-3) + (-4)\right]$.

      In a similar manner, have the pupils try:

      Add: +2, -8, +5; Add: -6, +8, -4

9. Have pupils notice that when we apply only the associative principle, we may not change the order of the addends.

B. Suggested Practice

1. Each of the following examples can be worked mentally if you first use the associative law.

   Rewrite each expression, using the associative law, and then find the sum mentally.

   a. $(17 + 25) + 75$

   b. $(123 + 125) + 175$

   c. $32 + (68 + 359)$

   d. $96 + (104 + 297)$

   e. $(3789 + 1250) + 1750$

2. Examples similar to III-A-8.

IV. Combining the Commutative and Associative Properties of Addition for the Numbers of Arithmetic (non-negative numbers)

A. Suggested Procedure

1. Review with pupils that the associative property of addition tells us that we may add three numbers by associating the middle number either with the preceding number or with the following number.

   Thus, to add 5, 8, and 4, we may associate 8 with 5 and their sum with 4, or we may associate 8 with 4 and their sum with 5.

   Then $(5 + 8) + 4 = 5 + (8 + 4)$

2. Have pupils now also use the commutative property to illustrate the addition of these three numbers. Each leads to the same sum.

3. Have pupils consider the following:

   a. $(5 + 8) + 4 = 13 + 4$, or 17

   b. $5 + (8 + 4) = 5 + 12$, or 17

   c. $5 + (4 + 8) = 5 + 12$, or 17

   d. $(5 + 4) + 8 = 9 + 8$, or 17

   e. $(4 + 5) + 8 = 9 + 8$, or 17

Methods a and b lead to the same result by associativity; methods b and c by commutativity; methods c and d by associativity; methods d and e by commutativity.

4. Guide pupils to see that the associative and commutative properties allow us to move addends around as we please in the process of finding a sum of several numbers. That is to say, they may be grouped or ordered in any way without affecting the result.

5. Have pupils see that without using addition facts we can show that

   $(5 + 8) + 4 = (5 + 4) + 8$

   We would proceed as follows:

$(5 + 8) + 4 = 5 + (8 + 4)$	Associative Property
$5 + (8 + 4) = 5 + (4 + 8)$	Commutative Property
$5 + (4 + 8) = (5 + 4) + 8$	Associative Property

   Therefore $(5 + 8) + 4 = (5 + 4) + 8$

   Note: This is considered a proof that the numerals $(5 + 8) + 4$ and $(5 + 4) + 8$ represent the same number, because we have used the properties of addition to show a series of equalities.

6. Have pupils use a similar procedure to show that

   $(5 + 8) + 4 = (4 + 8) + 5.$

7. The associative property of addition may be extended to four or more numbers.

B. Suggested Practice

Without using the addition facts, show that the following equalities are true because of the commutative and associative properties of addition. In each case, and for each step, indicate the property that justifies it.

1. $(3 + 6) + 9 = 3 + (6 + 9)$
2. $5 + 2 = 2 + 5$
3. $9 + 12 = 12 + 9$
4. $(56 + 17) + 15 = 15 + (17 + 56)$
5. $12 + (3 + 8) = (12 + 3) + 8$
6. $6 + (7 + 3) = 6 + (3 + 7)$
7. $(4 + 5) + 8 = (5 + 4) + 8$
8. $8 + (6 + 7) = (8 + 7) + 6$
9. $8 + (3 + 4) = (8 + 4) + 3$
10. $(5 + 6) + 3 = 5 + (3 + 6)$
11. $(8 + 6) + 2 = (8 + 2) + 6$

V. Combining the Commutative and Associative Properties of Addition for Signed Numbers

A. Suggested Procedure

1. Let us investigate to see whether the method we have chosen for adding signed numbers retains the properties we found to hold in adding the numbers of arithmetic.

2. The development parallels that for non-negative numbers as indicated in IV-A.

3. Have pupils realize that this method of adding signed numbers guarantees that the commutative and associative properties of addition for arithmetic numbers also hold for signed numbers.

B. Suggested Practice (Similar to IV-B)

VI. Multiplication of Signed Numbers

A. Suggested Procedure

1. Review meaning of absolute value.

   Introduce symbol for absolute value. The mathematical symbol for "absolute value of" is a pair of vertical bars enclosing the number. Thus, $|+5| = 5$ indicates that the absolute value of +5 is 5. Similarly,

$$|-5| = 5$$
$$0 = 0$$

2. Discuss with pupils various ways of expressing multiplication.

   a. $4 \times 5 \times 3$ (Using the "times" sign)

   b. $8 \cdot 9$ (Using the elevated dot)

   c. $(8)(9)$ (Using parentheses to indicate multiplication)

   d. $3x$ (Using no symbols)

   Note to teacher: Three approaches for developing multiplication of signed numbers are presented here. They are (1) the experimental approach, (2) the pattern approach and (3) the mathematical structure approach. All three lead to the definition of multiplication of signed numbers. They appear in increasing order of abstraction. Methods (1) and (2) are designed to make the definition of multiplication plausible to the students. Method (3) provides an informal proof or demonstration. Teachers will select the method or methods which they feel meet the needs of their pupils. Some will present two approaches, some all three, with (3) serving as the culmination.

3. Experiential approach to multiplication of signed numbers

   a. Multiplication of a positive number by a positive number

      1) Pose a problem: John will deposit four dollars each month for three months. How much money will he then have in his bank account?

      2) Elicit from pupils the agreement that: a bank deposit may be represented by a positive number.

         a withdrawal may be represented by a negative number,

         time in the future may be represented by a positive number,

         time in the past may be represented by a negative number.

      3) Elicit from pupils that three months from now, (+3), there will be \$12 more (+12) in John's bank account.

$$\begin{array}{r} +4 \\ +4 \\ \underline{+4} \\ +12 \end{array} \quad \text{or} \quad (+3)(+4) = +12$$

      4) Have pupils solve several similar problems and arrive at the understanding that we multiply positive numbers as we do the numbers of arithmetic. The product is always positive.

b. Multiplication of a negative number by a positive number

1) Pose problem: John plans to withdraw four dollars from his savings account each month for the next three months. What effect would that have on his bank account?

2) Elicit:

Four dollars withdrawn may be represented by -4.
Three months from now may be represented by +3.
Then, three months from now (+3), there will be $12 less (-12) in John's bank account.

$$\begin{array}{r} -4 \\ -4 \\ \underline{-4} \\ -12 \end{array} \quad \text{or } (+3)(-4) = -12$$

3) Have pupil solve several similar problems and arrive at the generalization that the product of a negative number and a positive number is the NEGATIVE of the product of their absolute values.

c. Multiplication of a positive number by a negative number

1) Pose problem: John deposited four dollars each month for three months. Did he have more money or less money in his bank account three months ago than he has now? How much more or less?

2) Elicit:

Four dollars deposited may be represented as +4; three months ago may be represented as -3. These facts may be represented by (-3)(+4). If he has deposited four dollars for three months, three months ago he must have had twelve dollars less (-12) than now. Therefore, (-3)(+4) = -12.

3) Have pupils solve several similar problems and arrive at the generalization that the product of a positive number by a negative number is the NEGATIVE of the product of their absolute values.

d. Multiplication of a negative number by a negative number

1) Pose problem: John has withdrawn four dollars each month for three months. Did he have more money or less money three months ago than he has now? How much more money did he have in his account three months ago?

2) Elicit:

Four dollars withdrawn may be represented by -4.
Three months ago may be represented by -3.
Three months ago he had $12 more (+12) in his account than he has now.

(-3)(-4) = +12

3) Have pupils solve several such problems and arrive at the generalization that the product of a negative number by a negative number is the product of their absolute values.

e. Have pupils solve a number of problems in mixed practice and help them arrive at the generalizations that, in multiplying signed numbers:

1) The product of two numbers with LIKE signs is a POSITIVE number.

2) The product of two numbers with UNLIKE signs is a NEGATIVE number.

4. Pattern approach to multiplication of signed numbers

a. Multiplying a positive number by a positive number

1) Elicit that positive numbers behave like the numbers of arithmetic.

2) Have pupils develop these two columns:

In Arithmetic	Signed Numbers
$(2)(5) = 10$	$(+2)(+5) = +10$
$(2)(4) = 8$	$(+2)(+4) = +8$
$(2)(3) = 6$	$(+2)(+3) = +6$

3) Elicit generalization: The product of a positive number by a positive number is positive.

b. Multiplying a negative number by a positive number

1) Have pupils consider the following:

$(+2)(+3) = +6$

$(+2)(+2) = +4$

$(+2)(+1) = +2$

$(+2)(0) = 0$ (As in arithmetic, the product of zero and any number is zero.)

2) Pose question: If we continue the pattern, what multiplicand will come after zero? (-1)

3) What is the product of (+2)(-1)?

Have pupils observe that whenever the multiplicand is decreased by 1, the product is decreased by 2.

Have pupils continue the pattern as indicated:

(+2)(-1) = ?   (-2)

(+2)(-2) = ?   (-4)

(+2)(-3) = ?   (-6)

(+2)(-4) = ?   (-8)

4) Elicit generalization: The product of a negative number by a positive number is the NEGATIVE of the product of their absolute values.

c. Multiplication of a positive number and a negative number

1) Have the pupils develop this sequence:

(+3)(+1) = +3

(+2)(+1) = +2

(+1)(+1) = +1

(0) (+1) = 0

(-1)(+1) = -1

(-2)(+1) = -2

(-3)(+1) = -3

2) Elicit the generalization: The product of a positive number and a negative number is the NEGATIVE of the product of their absolute values.

d. Multiplication of a negative number by a negative number

1) Have pupils develop this sequence:

(-2)(+4) = -8

(-2)(+3) = -6

(-2)(+2) = -4

(-2)(+1) = -2

(-2)(0) = 0

(-2)(-1) = +2

(-2)(-2) = +4

(-2)(-3) = +6

(-2)(-4) = +8

2) Elicit the generalization: The product of a negative number and a negative number is the product of their absolute values.

e. From these patterns, pupils are led to arrive at the generalization that in multiplication

1) the product of two numbers with like signs is a positive number,

2) the product of two numbers with unlike signs is a negative number.

Note: Method 3 - The mathematical structure approach is shown on page 27.

B. Suggested Practice

Multiply as indicated:

1. (+2)(-7)
2. (+2.5)(-.3)
3. (+6)(+8)
4. (-12)(-7)
5. (-6)(36.5)
6. (+7)(-5)
7. $(-\frac{2}{3})(+\frac{2}{7})$
8. (-4)(+15)
9. $(+\frac{1}{2})(-2)$
10. (-4)(-2)(+3)

VII. Commutative and Associative Properties of Multiplication for Signed Numbers

A. Suggested Procedure

1. Develop the commutative property of multiplication for signed numbers.

a. Review with pupil the commutative property of addition with signed numbers. The order of the addends may be changed without affecting the sum.

b. Pose problem: Does the commutative property hold for the multiplication of signed numbers?

c. Have pupils develop illustrations of the three possible combinations of positive and negative numbers as indicated:

1) (+3)(+6) = +18
(+6)(+3) = +18
$\therefore$ (+3)(+6) = (+6)(+3)

2) (+7)(-6) = -42
(-6)(+7) = -42
$\therefore$ (+7)(-6) = (-6)(+7)

3) $(-4)(-3) = +12$

$(-3)(-4) = +12$

$\therefore (-4)(-3) = (-3)(-4)$

4) Have pupils make up several additional examples illustrating the commutative principle for the multiplication of signed numbers.

Note to teacher: If the pattern approach was used, the assumption that the pattern continues implies that the properties of the operations of arithmetic (commutation, association) will still hold for those operations on signed numbers. The above examples illustrate this.

If the experiential approach is used solely, the pupil should be led to see that the definitions we have adopted for the multiplication of signed numbers result in retaining the commutative and associative properties for signed numbers.

2. Develop the associative property of multiplication for signed numbers.

a. Review with pupils the fact that the fundamental operations of arithmetic are binary operations, that is, they are performed on two numbers.

b. Have pupils give illustrations of the fact that we often have occasion to multiply three numbers. Pose the question: How can we multiply three numbers when multiplication is a binary operation?

c. Elicit that when we multiply 3, 5, and 6, we are really performing the operation of multiplication twice, as follows:

$3 \times 5 = 15$ and $15 \times 6 = 90$

d. Have pupils observe that we first found the product of 3 and 5, and then multiplied it by 6. Stated symbolically, our procedure was as follows:

$(3 \times 5) \times 6 = 15 \times 6 = 90$

e. Elicit that we may obtain the same result by first multiplying 5 and 6, obtaining 30, and then multiplying 30 by 3.

$3 \times (5 \times 6) = 3 \times 30 = 90$

$\therefore (3 \times 5) \times 6 = 3 \times (5 + 6)$

f. Have pupils repeat this procedure with several examples.

g. Guide pupils to generalize that the manner in which the factors are grouped does not affect the final product.

h. Lead pupils to observe that this property of multiplication for the numbers of arithmetic is analogous to the associative property of addition. It is called the associative property of multiplication and is frequently referred to as the associative law or associative principle of multiplication.

It must be noted that when we apply only the associative property of multiplication, the order of the factors is not changed.

i. Now pose this question: Does the associative property of multiplication hold for signed numbers?

j. Guide pupils to answer by having them consider the following:

1) Multiply 4, 8, and 9

This may be grouped: $(4 \times 8) \times 9 = 32 \times 9 = 288$

or $4 \times (8 \times 9) = 4 \times 72 = 288$

$\therefore (4 \times 8) \times 9 = 4 \times (8 \times 9)$

2) Multiply -4, -8, and -9

Similarly: $(-4 \times -8) \times (-9) = 32 \times (-9) = -288$

or $-4 \times (-8 \times -9) = -4 \times 72 = -288$

$\therefore (-4 \times -8) \times (-9) = -4 \times (-8 \times -9)$

3) Multiply -2, +5, and +6.

4) Multiply +3, -6, and -7.

Similarly: $(+3 \cdot -6) \cdot -7 = -18 \cdot -7 = +126$

or $+3 \cdot (-6 \cdot -7) = +3 \cdot +42 = +126$

$\therefore (+3 \cdot -6) \cdot -7 = +3 \cdot (-6 \cdot -7)$

k. Elicit the generalization that on the basis of the examples shown, we may assume that the associative property applies to multiplication for signed numbers.

3. Combining the commutative and associative properties of multiplication for signed numbers.

a. Have pupils suggest several different ways of finding the product $-4 \cdot +6 \cdot -5$. They may suggest:

1) $-4 \cdot (+6 \cdot -5) = -4 \cdot -30$, or 120

2) $(-4 \cdot +6) \cdot -5 = -24 \cdot -5$, or 120

3) $(+6 \cdot -4) \cdot -5 = -24 \cdot -5$, or 120

4) $+6 \cdot (-4 \cdot -5) = +6 \cdot +20$, or 120

Illustrations 1) and 2) give the same product by associativity, illustrations 2) and 3) by commutativity, and illustrations 3) and 4) by associativity.

b. Elicit the generalization that when we multiply signed numbers, they may be grouped or ordered in any way whatsoever without affecting the result. Therefore, grouping symbols may be omitted in indicating the product of signed numbers because the result is unique.

c. Have pupils see that without using multiplication facts we may show that the numbers represented by $6 \times (4 \times 5)$ and $(4 \times 6) \times 5$ are the same. We would proceed as follows:

Show that $6 \times (4 \times 5) = (4 \times 6) \times 5$

$6 \times (4 \times 5) = (6 \times 4) \times 5$ Associative Property

$= (4 \times 6) \times 5$ Commutative Property

d. Have pupils use a similar procedure to show that

$6 \times (4 \times 5) = (6 \times 5) \times 4$

e. The associative property of multiplication may be extended to four or more numbers.

$7 \times 2 \times 5 \times 3 = (7 \times 2 \times 5) \times 3$ or

$7 \times (2 \times 5 \times 3)$ or

$(7 \times 2) \times (5 \times 3)$, and so on.

B. Suggested Practice

Without using multiplication facts, have pupils show that the following equalities are true because of the commutative and/or associative properties of multiplication.

1. $5 \times 9 = 9 \times 5$
2. $4 \times (\frac{1}{2} \times 3) = 4 \times (3 \times \frac{1}{2})$
3. $(3 \times 2) \times 6 = 3 \times (2 \times 6)$
4. $-4 \times (-2 \times -6) = (-4 \times -2) \times -6$
5. $+1 \cdot -7 = -7 \cdot +1$
6. $+.8 \times (-1.1 \times 3.2) = (+.8 \times -1.1) \times 3.2$

7. $\frac{1}{2} \times \frac{3}{4} = \frac{3}{4} \times \frac{1}{2}$

8. $5 \times (3 \times 4) = (3 \times 5) \times 4$

9. $(12 \times 7) \times 3 = (3 \times 7) \times 12$

10. $(-6 \cdot -4) \cdot 2 = (-2 \cdot -4) \cdot -6$

VIII. The Distributive Property of Multiplication Over Addition for the Set of Signed Numbers

A. Suggested Procedure

1. Review the distributive property for the numbers of arithmetic.

a. $3 \times 35 = 3 \times (30 + 5)$

$= (3 \times 30) + (3 \times 5)$

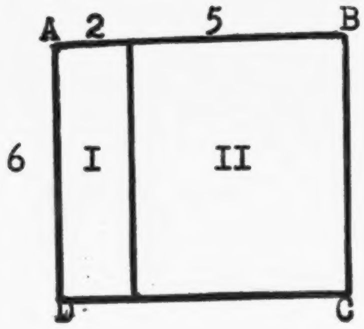
$= 90 + 15$, or 105

$8 \times 4\frac{1}{2} = 8 \times (4 + \frac{1}{2})$

$= (8 \times 4) + (8 \times \frac{1}{2})$

$= 32 + 4$, or 36

b. Pose problem: How can we find the area of this rectangle?



Method 1

$A = w\ell$, where $w = 6$ and $\ell = 2 + 5$

$A = 6\ (2 + 5) = 6 \times 7$ or 42 square units

Method 2

A = sum of areas of rectangles I and II

$A = (6 \times 2) + (6 \times 5) = 12 + 30$ or 42 square units

Have pupils observe that both methods give the same result because

$6\ (2 + 5) = (6 \times 2) + (6 \times 5)$

c. Have pupils try several area problems of the same type.

d. Guide them to the generalization that the product of a number and the sum of two numbers may be found in two ways:

1) Obtain the sum first and then multiply:

$13 \times (10 + 2) = 13 \times 12 = 156$ or

2) Multiply each addend by the multiplier and then find the sum of these products:

$13 \times (10 + 2) = (13 \times 10) + (13 \times 2)$

$= 130 + 26 = 156$

e. Inform pupils that the second method is an application of the Distributive Property of Multiplication Over Addition. The word "distributive" suggests that when you have a product of a multiplier and a sum, you may distribute or spread the multiplier, using it once on each addend.

2. Since in our examples we have dealt only with positive numbers, have pupils try examples with signed numbers to see whether this distributive principle holds for the enlarged set of numbers.

Note: In these examples, brackets are used as grouping symbols. The parentheses are used only to make clear the sign of the number.

a. $8 \times [(-4) + (-5)] = 8 \times (-9) = -72$ or

$8 \times (-4) + 8 \times (-5) = (-32) + (-40) = -72$

$\therefore 8 \times [(-4) + (-5)] = 8 \times (-4) + 8 \times (-5)$

The teacher may demonstrate with other numbers.

b. $(-4) \times [(-12) + (-4)] = (-4) \times (-16) = +64$ or

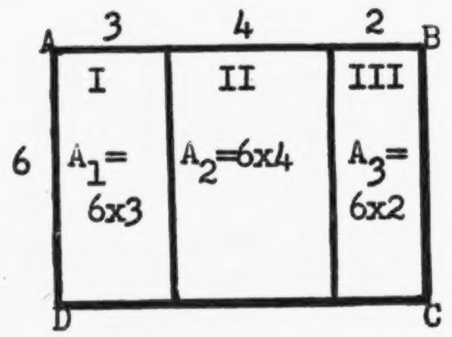
$(-4) \times (-12) + (-4) \times (-4) = (+48) + (+16) = +64$

$\therefore (-4) \times [(-12) + (-4)] = (-4) \times (-12) + (-4) \times (-4)$

The teacher may demonstrate with other numbers.

c. Elicit the generalization that on the basis of the examples, we may assume that the distributive property of multiplication over addition applies to the set of signed numbers.

5. Elicit that the distributive property may be applied to three or more addends, as illustrated.



A = 6 x (3 + 4 + 2) = 6 x 9 = 54 square units

A = (6 x 3) + (6 x 4) + (6 x 2)

18 + 24 + 12 = 54 square units

B. Suggested Practice

1. Fill in the blanks below so that each statement is an illustration of the distributive law.

a. 6· (3 + 2) = _____

b. _____ = 9 x 2 + 9 x 8

c. _____ = 10 · 6 + 10 · 2

d. 7· (   ) = _____ ·3 + _____ · 4

e. _____ ·(6 + 8) = 3· _____ + 3 · _____

2. Use the distributive law to obtain the value of each numerical expression.

a. 3 x 6 + 3 x 12	f. $\frac{1}{2}$ x 4 + $\frac{1}{2}$ x 8 + $\frac{1}{2}$ x 12
b. 9 x 8 + 9 x (-2)	g. 19 x $\frac{3}{4}$ + 19 x $\frac{1}{4}$
c. 37 x 93 + 37 x 2	h. 3 · -4 + 3 · -2 + 3 · + 6
d. 25 x 7 + 25 x (-3)	i. 42 x 106 = 42 x (100 + 6) = ?
e. 4 x 5 + 4 x 3 + 4 x 2	j. 25 x 45 = 25 x (40 + 5) = ?

3. In each of the following, state the property illustrated.

   a. $(3 \times 8) \times 7 = 3 \times (8 \times 7)$

   b. $25 + 16 = 16 + 25$

   c. $-15 \times 30 = 30 \times -15$

   d. $6 + (4 + 7) = (6 + 4) + 7$

   e. $(-5 \times 9) + (-5 \times 10) = -5 \times (9 + 10)$

   f. $16 \times (13 + 12) = 16 \times 13 + 16 \times 12$

   g. $-45 \times (16 \times 5) = (-45 \times 16) \times 5$

   h. $-71 \times (180 + 20) = (-71 \times 180) + (-71 \times 20)$

   i. $(33 + 18) + 4 = 33 + (18 + 4)$

   j. $103 \times 2 + 103 \times 8 = 103 \times (2 + 8)$

4. Using the principles of associativity, commutativity, and distributivity, establish the truth of the following equalities without using the facts of multiplication and addition.

   a. $2 \times (4 + 7) = 4 \times 2 + 7 \times 2$

   b. $-3 \times (2 + 4) = -3 \cdot 2 + 4 \cdot -3$

   c. $48 \cdot (\frac{3}{4} + \frac{1}{2}) = 48 \cdot \frac{1}{2} + 48 \cdot \frac{3}{4}$

   d. $2 \times 5 + 8 \times 2 = 2 \times (8 + 5)$

   e. $(2 \cdot 4 + 3 \cdot 5) + 6 \cdot 4 = 4\ (2 + 6) + 5 \cdot 3$.

C. Mathematical Structure Approach to Multiplication of Two Negative Numbers (See page 20)

Procedure: Explain to the class that the definitions of the multiplication of signed numbers for various combinations of signs can be developed by choosing these definitions in such a way that the results of the operations on the numbers of arithmetic are preserved and that the commutative, associative, and distributive laws hold for the signed numbers.

First consider the product of two positive numbers such as $(+3)(+2)$. Elicit that positive numbers correspond to the numbers of arithmetic, and hence the product $(+3)(+2)$ should correspond to the product $(3)(2)$. Hence, $(+3)(+2) = +6$. Elicit the generalization for the product of any two positive numbers. Will the commutative and associative laws hold for the products of positive numbers? Why?

Next, consider the product of a negative number by a positive number, for example, $(+3)(-2)$. This may be interpreted as $3(-2)$ or the sum of three addends, each equal to -2. What is this sum? Elicit the generalization for the product of a negative number by a positive number.

Third, consider the product of a positive number by a negative number, for example, $(-2)(+3)$. What principle will relate this example to the previous one? Therefore what should the product be if we want this principle to hold? Elicit the generalization for the product of numbers of unlike signs.

Finally, consider the product of two negative numbers, e.g., $(-3)(-2) = ?$

$(-3)\ (0)$	$= 0$	The product of 0 and any number is 0.
$(-3)\left[2 + (-2)\right]$	$= 0$	The sum of a number and its opposite is 0.
$(-3)(-2) + (-3)(-2)$	$= 0$	Distributive property
$-6 + (-3)\ (-2)$	$= 0$	Have pupils discuss what the product $(-3)(-2)$ must be if the sum of -6 and that product is to equal 0. (From our idea of opposites, $(-3)(-2)$ must equal +6.)

IX. The Subtraction of Signed Numbers

Note: For many years pupils used the "-" sign to indicate the operation of subtraction. In Grade 8, they developed an understanding of the use of this sign to indicate direction. The concept of "opposites" was introduced, but now the symbol to show opposites, namely the "-", should be presented. Thus, $-(-2)$ is read "the opposite of negative two."

As with multiplication, various approaches to the teaching of subtraction of signed numbers are presented, so that the teacher may select the approach or approaches deemed most suitable for a particular class of pupils. These approaches are used to make the definition of subtraction of signed numbers appear reasonable when it is finally formulated.

A. Suggested Procedure

1. Have pupils review:

    a. Meaning of signed numbers
    b. Addition of signed numbers
    c. Opposites
    d. Inverse operations
    e. Number line
    f. The order of signed numbers (the convention that a number "to the right" of another on the number line is the larger of the two numbers)
    g. Signs of operation
    h. Signs of direction

2. Approach to Subtraction through the Number Line

a. Addition and subtraction are inverse operations

1) Have pupils consider the operation of subtraction, e.g., 6 - 4. Have them observe that when we perform this subtraction, we are finding a number which when added to 4 will give 6. That is to say,

$6 - 4 = 2$ because $4 + 2 = 6$

2) Have pupils write the addition examples that correspond to the following subtraction examples:

$8 - 5 = 3$ because $5 + 3 = 8$

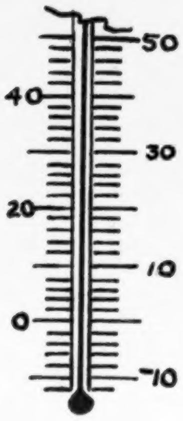
$14 - 7 = 7$

$12 - 8 = 4$

Have pupils notice that to perform subtraction, addition is used. Addition and subtraction are said to be inverse operations.

b. A Thermometer Scale

Temperatures above zero are designated "+" while those below zero are "-". A rise in temperature is a change in a positive direction, a drop is a change in a negative direction.



1) Pose problem: One morning the thermometer read 35° above zero. By afternoon it read 45° above zero. What was the change in temperature? How would you represent this change as a signed number?

Solution: To obtain the answer, we subtract the earlier reading 35° above zero (+35), from the later reading, 45° above zero (+45). We record this:

$(+45) - (+35) = ?$ or

$$\begin{array}{r}(+45)\\ -\underline{(+35)}\\ ?\end{array}$$

Begin at +35 and count to +45. How many units have we counted and in what direction? We moved 10 units in the positive direction. This means that +10 must be added to +35 to get +45. Then, (+45) - (+35) = +10

2) Pose problem: On a cold January afternoon the thermometer read 5° above zero. That evening it read 2° below zero. How much had the temperature changed? Represent the amount of change as a signed number.

Solution: To obtain the answer, we subtract the earlier reading, 5° above zero (+5), from the later reading, 2° below zero (-2). We record this:

$(-2) - (+5) = ?$ or $\begin{array}{r} (-2) \\ -\underline{(+5)} \\ ? \end{array}$

Then begin at +5 and count to -2. How many units have we counted and in what direction? We moved 7 units in the negative direction. This means that -7 must be added to +5 to get -2. Therefore, (-2)-(+5)= -7.

3) Have pupils solve similar problems involving a thermometer scale.

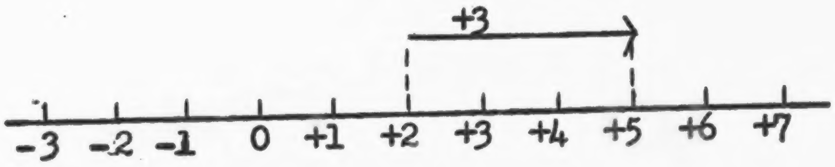
c. Have pupils draw a number line to illustrate each of the following:

It is agreed that movement to the right will be a positive direction and movement to the left will be a negative direction. (In each of the illustrations the starting point will be zero.)

1) (+5) - (+2) = ?

This means finding what number I must add to (+2) to arrive at (+5).

Think: "I am at a point two units to the right of zero. What motion must be made to reach the position 5 units to the right of zero?"

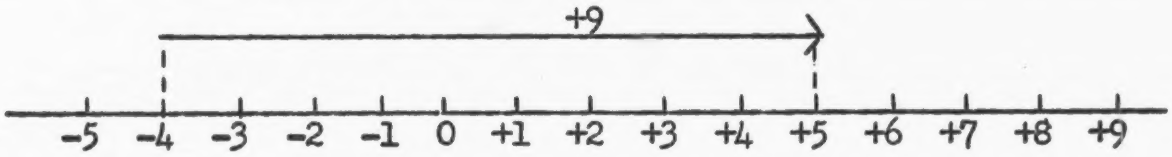


From +2 it is necessary to go 3 units in a positive direction to get to +5. Hence (+5) - (+2) = +3 or $\begin{array}{r} (+5) \\ -\underline{(+2)} \\ +3 \end{array}$ or subtract $\begin{array}{r} (+5) \\ \underline{(+2)} \\ +3 \end{array}$

2) (+5) - (-4) = ?

This means finding what number I must add to (-4) to arrive at (+5).

Think: "I am at a point 4 units to the left of zero. What motion must be made to reach the position 5 units to the right of zero?"

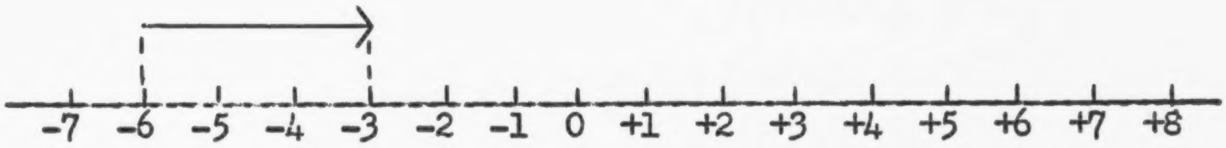


From -4 it is necessary to go 9 units in a positive direction to get +5. Hence (+5) - (-4) = +9 or $\begin{array}{r}(+5)\\ -(\underline{-4})\\ ?\end{array}$ or subtract $\begin{array}{r}(+5)\\ \underline{(-4)}\\ ?\end{array}$

3) (-3) - (-6) = ?

This means finding what number I must add to (-6) to arrive at (-3).

Think: "I am at a point 6 units to the left of zero. What motion must be made to reach the position 3 units to the left of zero?

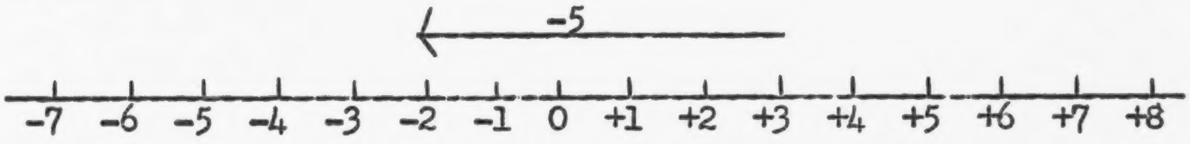


From -6 it is necessary to go 3 units in a positive direction to get to -3. Hence (-3) - (-6) = ? or $\begin{array}{r}(-3)\\ -(\underline{-6})\\ ?\end{array}$ or subtract $\begin{array}{r}-3\\ \underline{-6}\\ ?\end{array}$

4) (-2) - (+3) = ?

This means what number must I add to (+3) to arrive at (-2)?

Think: "I am at a point 3 units to the right of zero. What motion must be made to reach the position 2 units to the left of zero?"



From +3 it is necessary to go 5 units in a negative direction. Hence, (-2) - (+3) = ? or $\begin{array}{r}(-2)\\ -(\underline{+3})\\ ?\end{array}$ or subtract $\begin{array}{r}-2\\ \underline{+3}\\ ?\end{array}$

5) In a similar manner, using the number line, develop the following examples in the subtraction of signed numbers:

a) $(+5) - (0) = ?$
c) $(-6) - (-6) = ?$
e) $(-8) - (+4)$
b) $(0) - (-3) = ?$
d) $(+6) - (-5) = ?$
f) $(-5) - (-3)$

6) Discuss with pupils the impracticability of using the number line for subtraction if the numbers are large.

Have pupils use the number line to find the answers to the following subtraction examples:

Set I

$(+4) - (+9) = -5$
$(-4) - (+6) = -10$
$(+5) - (-3) = +8$
$(-2) - (-1) = -1$

Have pupils now find the answers to the following addition examples:

Set II

$(+4) + (-9) = -5$
$(-4) + (-6) = -10$
$(+5) + (+3) = +8$
$(-2) + (+1) = -1$

d. Guide pupils' thinking as follows:

1) Compare the operations in Set I and Set II.

2) Compare the answers.

3) Compare the second numbers of the corresponding examples in Sets I and II.

How can a subtraction example be converted to an addition example?

On the basis of these and similar examples, pupils conclude that the subtraction of a number is equivalent to the addition of its opposite.

e. Have pupils illustrate the foregoing subtraction principle by writing each of the following subtraction examples as an addition example:

Subtraction	Addition	Value
$+(5) - (+3)$	$(+5) + (-3)$	$+2$
$+(5) - (+10)$		
$+(12) - (-4)$		
$(-2\frac{1}{2}) - (+3\frac{1}{4})$		
$(+6.4) - (-2.8)$		

4. Approach to Subtraction through Equations

   a. Have pupils review:

      1) Meaning of a variable

      2) Expressing a problem situation as an open sentence (equation)

      3) Solving equations by the addition principle; by the subtraction principle

   b. Have pupils consider how a cashier makes change. If the purchase price is 72 cents and the cashier is given a dollar bill, the amount of change, or difference, is found by adding to the purchase price until a dollar is reached.

      Thus, to subtract 72 from 100, we find the number which must be added to 72 to obtain 100.

      The question $100 - 72 = ?$ means the same as $72 + ? = 100$ or $72 + x = 100$ in equation form.

   c. Have pupil observe that the equation $72 + x = 100$ may be solved as follows:

Subtraction Principle	Addition of Opposite
$72 + x - 72 = 100 - 72$	$72 + x + (-72) = 100 + (-72)$
$x = 100 - 72$	$x = 100 + (-72)$

   Guide pupils to see that $100 - 72$ and $100 + (-72)$ represent the same number, namely, 28.

   d. Have pupils use the above observation to express the following subtractions as corresponding additions: (check each)

	Check	
$15 - 7$ means $15 + (-7)$	$15 - 7 = 8$	$15 + (-7) = 8$
$10 - (+9)$		
$12 - (+6)$		
$11\frac{1}{2} - (+2)$		

e. Guide pupil to generalize that subtracting a positive number from a larger positive number gives the same result as adding the opposite.

Will this method apply to the subtraction of any numbers?

f. Develop principles for subtraction of signed numbers.

1) Have pupil consider $(+20) - (+6) = ?$

This means $(+6) + (?) = +20$, or $(+6) + x = (+20)$

Solve $(+6) + x = (+20)$ by adding the opposite.

$(+6) + x + (-6) = (+20) + (-6)$

$x = (+20) + (-6)$

Thus, $(+20) - (+6) = (+20) + (-6)$ or $+14$

2) Have pupils consider $(+7) - (-2) = ?$

This means $(-2) + ? = (+7)$ or $(-2) + x = (+7)$

Solve $(-2) + x = (+7)$ by adding the opposite.

$(-2) + x + (+2) = (+7) + (+2)$

$x = (+7) + (+2)$

Thus, $(+7) - (-2) = (+7) + (+2)$ or $+9$

3) Have pupils consider $(+7) - (+7) = ?$

This means $(+7) + ? = (+7)$ or $(+7) + x = (+7)$

Solve $(+7) + x = (+7)$ by adding the opposite.

$(+7) + x + (-7) = (+7) + (-7)$

$x = (+7) + (-7)$

Thus, $(+7) - (+7) = (+7) + (-7) = 0$

4) Have pupils solve several similar examples.

5) An examination of these examples should lead to the principle that subtracting a number is the same as adding its opposite. Hence, every subtraction problem can be solved as an addition problem. The principle of opposites may be used in each case of subtraction of numbers.

6) Have pupils illustrate the subtraction principle by writing each of the following subtraction examples as an addition example.

Subtraction	Addition	Value
(+4) - (+11)	(+4) + (-11)	-7
(+16) - (+9)		
(+6) - (-10)		
($-\frac{1}{2}$) - ($+\frac{1}{2}$)		
(+12.3) - (+12.2)		

7) Discuss with pupils the meaning of a problem such as

2 - 7 = ?

Does the sign (-) mean subtract or does it mean that the second term of the expression is (-7)?

Guide pupils to see that the sign (-) in 2 - 7 means "subtract". Otherwise there is no sign which tells us what operation to perform. So the problem "2 - 7" is equivalent to the problem "(+2) - (+7)".

However, the subtraction rule leads us immediately to the form (+2) + (-7).

And so, when confronted with the problem 2 - 7, one usually thinks

(+2) and (-7) give (-5)

This principle can be extended to apply to cases such as:

2 - 3 + 5 - 8 - 7 = 2 + (-3) + (+5) + (-8) + (-7)

The commutative and associative principle may then be applied to facilitate the computation, if desirable.

B. Suggested Practice

1. Subtract:

a. +8	b. -5	c. -4	d. +9	e. 0
+12	+3	-8	-3	-6

2. Perform the indicated operation:

a. (+18)	b. (-10)	c. (+17)	d. (-9)	e. (+5)
-(+ 6)	-(-10)	-(+20)	-(+3)	-(-2)
f. (-3.5)	g. (+6 7/10)	h. (-4.1)	i. (+7 3/4)	j. (-8 4/5)
-(+4.72)	-(-3 2/5)	-(-9.3)	-(+9 3/8)	-(+3 1/2)

3. Subtract:

   a. $(+5) - (-8) = ?$

   b. $(+\frac{1}{2}) - (-1) = ?$

   c. $(-3.02) - (-.6) = ?$

   d. $(+24) - (-35) = ?$

   e. $(-687) - (-786) = ?$

4. Subtract +6 from +18.

5. From +5 take -6.

6. Find the remainder when -8 is subtracted from +17.

   Solve these problems by using signed numbers.

7. On the hottest day of one summer the temperature was 98°.
   On the coldest day that year the temperature was (-9°).
   What was the change in temperature from the coldest to the hottest day?

8. What change must occur to raise the temperature from (-10°) to 34°?

9. If New York's latitude is 41°N and that of Buenos Aires is 34°S, what is the difference in latitude between the two cities?

10. What is the change in altitude in going from Jerimoth Hill, Rhode Island, 812 feet above sea level to New Orleans, Louisiana, 5 feet below sea level?

11. Your watch is 6 minutes faster than the radio time. The clock in your school is 2 minutes slower than your watch. Is the clock in your school faster or slower than the radio time? By how much?

X. Properties of Subtraction

A. Suggested Procedure

1. Is there a commutative property of subtraction?

   a. Review the meaning of the commutative property of addition.

      1) $5 + 2 = 7$ $\quad 2 + 5 = 7$
         Therefore, $5 + 2 = 2 + 5$

      2) $(-8) + (-4) = -12$ $\quad (-4) + (-8) = -12$
         Therefore, $(-8) + (-4) = (-4) + (-8)$

   b. Is there a commutative property of subtraction?

      1) Does $5 - 2 = 2 - 5$?

         $5 - 2 = 3$ $\quad 2 - 5 = -3$ $\quad 3 \neq -3$ $\quad$ Therefore, $5 - 2 \neq 2 - 5$

2) Does $(-8) - (-4) = (-4) - (-8)$?

$(-8) - (-4) = -4$ $(-4) - (-8) = 4$ $-4 \neq 4$

Therefore, $(-8) - (-4) \neq (-4) - (-8)$

Pupils conclude there is no commutative property of subtraction.

Note: One counter-example would have been sufficient to establish this.

2. Is there an associative property of subtraction?

a. Review the meaning of the associative property of addition.

1) $(13 + 8) + 4 = 21 + 4$, or 25 $13 + (8 + 4) = 13 + 12$, or 25

Therefore, $(13 + 8) + 4 = 13 + (8 + 4)$

2) $\left[(-12) + (+2)\right] + (+4) = (-10) + (+4) = -6$ $(-12) + \left[(+2) + (+4)\right] =$

$(-12) + (+6) = -6$

Therefore, $\left[(-12) + (+2)\right] + (+4) = (-12) + \left[(+2) + (+4)\right]$

b. Is there an associative property of subtraction?

1) Does $(13 - 8) - 4 = 13 - (8 - 4)$?

$(13 - 8) -4 = 5 - 4 = 1$ $13 - (8 - 4) = 13 - 4 = 9$ $1 \neq 9$

Therefore, $(13 - 8) - 4 \neq 13 - (8 - 4)$

2) Does $\left[(-12) - (+2)\right] - (+4) = (-12) - \left[(+2) - (+4)\right]$?

$\left[(-12) - (+2)\right] - (+4) = (-14) - (+4) = -18$

$(-12) - \left[(+2) - (+4)\right] =$
$(-12) - (-2) = -10$
$-18 \neq -10$

Therefore, $\left[(-12) - (+2)\right] - (+4) \neq (-12) - \left[(+2) - (+4)\right]$

Pupils conclude there is no associative property of subtraction.

3. Is there a distributive property of multiplication over subtraction?

a. Review the distributive property of multiplication over addition.

1) $2\ (8 + 5) = 2 \cdot 13$, or 26 $2 \cdot 8 + 2 \cdot 5 = 16 + 10$, or 26

Therefore, $2(8 + 5) = 2 \cdot 8 + 2 \cdot 5$.

2) $4\left[(+9) + (-3)\right] = 4\ (+6) = 24 \qquad 4\ (+9) + 4\ (-3) = 36 + (-12) = 24$

Therefore, $4\left[(+9) + (-3)\right] = 4\ (+9) + 4\ (-3)$

b. Is there a distributive property of multiplication over subtraction?

1) Does $2\ (8 - 5) = 2 \cdot 8 - 2 \cdot 5$?

$2\ (8 - 5) = 2 \cdot 3$, or 6 $\qquad 2 \cdot 8 - 2 \cdot 5 = 16 - 10$, or 6

Therefore, $2\ (8 - 5) = 2 \cdot 8 - 2 \cdot 5$

2) Does $4\left[(+9) - (-3)\right] = 4\ (+9) - 4\ (-3)$?

$4\left[(+9) - (-3)\right] = 4(12) = 48 \qquad 4\ (+9) - 4\ (-3) = 36 - (-12) = 48$

Therefore, $4\left[(+9) - (-3)\right] = 4\ (+9) - 4\ (-3)$.

After several similar examples, the pupil will be led to the conclusion that the distributive property of multiplication over subtraction holds.

Have pupils realize that since subtraction has been defined as the addition of the additive inverse, any subtraction example can be expressed as a corresponding addition example. Inasmuch as the distributive property has already been established for addition, it holds for subtraction as well.

Thus, $2x\ (12 - 4)$ may be expressed as $2x\left[(2 + (-4)\right]$

$= 2 \times 12 + 2\ (-4)$

$= 24 - 8$ or 16

B. Suggested Practice (Similar to X-A)

XI. Division of Signed Numbers

A. Suggested Procedure

1. Review definitions developed in addition, subtraction, and multiplication of signed numbers.

2. Discuss the definition of division. Division is defined as the inverse operation of multiplication, in the same manner as subtraction is the inverse operation of addition. For example, consider the problem, "What is 8 divided by 2?"

   This asks the same question as, "By what number must 2 be multiplied to give 8?"

   The dividend is 8, the divisor is 2, and the answer is called the quotient, as in arithmetic. Zero may not be used as a divisor.

The division sign (÷) is seldom used in algebra. The problem, 10 divided by 2, or 10 ÷ 2 is more often written with the help of a fraction line as follows: $\frac{10}{2}$. Since 2 must be multiplied by 5 to obtain the product, 10, we have $\frac{10}{2} = 5$. Because of the use of the fractional form, the word numerator is used for the dividend, and the word denominator for the divisor.

3. Have pupils extend the definition of division (as the inverse of multiplication) to signed numbers.

Division		Multiplication	Answer
$\frac{+15}{+5}$	means	$(?)(+5) = +15$	+3
$\frac{-12}{+4}$	means	$(?)(+4) = -12$	-3
$\frac{-10}{-2}$	means	$(?)(-2) = -10$	+5
$\frac{+6}{-2}$	means	$(?)(-2) = +6$	-3

Pupils observe from a consideration of the signs of the answers to the foregoing examples, that the signs follow the rules for the multiplication of signed numbers.

4. On the basis of these developments, have pupils formulate the following rules for the division of signed numbers:

   a. The absolute value of an indicated division of numbers is the quotient of the absolute values of the numbers.

   b. The sign of a quotient follows the rule of signs when multiplying signed numbers.

B. Suggested Practice

1. Perform the indicated operations:

$\frac{(-36)}{-4}$

$\frac{(-42)}{7}$

$\frac{(+36)}{(-9)}$

$\frac{-24}{6} = ?$

$\frac{-60}{-20} = ?$

$\frac{+18}{+3} = ?$

$\frac{(+24)}{-6}$

$-8\overline{)-32}$

$(-33) \div (+3)$

$(-\frac{1}{4}) \div (-\frac{1}{2})$

$(-50) \div (+2)$

$(-2) \div (-1)$

2. Perform the indicated divisions:

a. $\frac{+48}{+12}$

b. $\frac{-72}{-12}$

c. $\frac{-4}{+16}$

d. $\frac{-8}{+16}$

e. $\frac{-250}{-50}$

f. $\frac{-4}{+20}$

g. $-6\overline{)+18}$

h. $\frac{+1.8}{9}$

i. $\frac{+28}{+.7}$

j. $\frac{-4.2}{-\ .2}$

k. $\frac{-5}{6} \div 5$

l. $+\frac{12}{7} \div (-4)$

m. $-6 \div (+\frac{1}{2})$

n. $\frac{-7}{4} \div 2\frac{1}{4}$

3. Divide:

a. 24 by -8

b. -36 by -20

c. 100 by -20

d. 100 by $\frac{1}{4}$

e. $3\frac{1}{3}$ by $2\frac{1}{6}$

f. -1.6 by 4

g. +1.44 by +1.2

h. 6.4 by .08

4. Perform the indicated divisions:

a. $\left[(+4)(+3)\right] \div (+2)$

b. $\left[(-9)(+4)\right] \div (+12)$

c. $\left[(-2.5)(-4)\right] \div (-5)$

d. $\left[(+2)(+\frac{3}{4})\right] \div (-2)$

e. $\left[(+\frac{1}{4})(-\frac{1}{2})\right] \div (-\frac{1}{8})$

f. $(+12) \div \left[(+3)(+2)\right]$

C. Using Directed Numbers to Find an Average (optional):

1. Have pupils compute an average.

   Problem: Jack received marks of 90, 85, 90, 95 on a series of mathematics tests. What is his average mark?

2. Ask pupils to study the numbers and see whether they can find the average in a shorter way.

   a. Have pupils consider what the average 90 implies. If everyone of the numbers was the same number, that number would be 90.

   b. Have them note that the numbers 85 and 95 each deviates by 5 from the average (90).

   c. Elicit that instead of adding and dividing, they can pair the excess of 5 in 95 with the insufficiency of 5 in 85, getting 90 in each instance as a result.

      Thus, for the purposes of obtaining an average, 90 85 90 95 may be considered as 90 90 90 90

3. Have pupils see that the concept of deviation can be used to find the average (90).

   a. Have pupils express the deviations of 85 and 95 from 90 by signed numbers.

      95 deviates from 90 by +5 and
      85 deviates from 90 by (-5)

   b. Set up a table:

Test Marks		Deviations from average (90)
90		0
85		-5
90		0
95		+5
	Sum of deviations	0

   c. The sum of the deviations of the score from the average is zero. Therefore, the average of the marks is 90 as they found in 1.

4. Pose question: Suppose we did not have the average as at the beginning. Would it be possible to find the average by the use of deviations of the scores from some assumed average? Guide them to see that they can use the concept of deviations to find the average.

a. Have pupils study the numbers and choose a reasonable assumed average, e.g., 87.

b. Have them set up a table as in 3b.

Test Marks	Deviations from Assumed Average (87)
90	+3
85	-2
90	+3
95	+8
Sum of deviations	+12

c. Have them consider:

What is the relation of +12 to the assumed average?

(+12) is the sum of all the deviations. Therefore, the average deviation is (+12) ÷ 4, or (+3). Then 87, the assumed average, plus (+3) does give the true average, 90.

The pupils see that by dividing the sum of the deviations by the number of deviations, they find the average deviation, which when added to the assumed average will give the true average.

Assumed average plus the average deviation is true average.

$$87 \quad + \quad (+3) \quad = \quad 90$$

d. Guide pupils to see that they can assume any average and obtain the same result. For the same scores assume an average of 92.

Test Marks	Average Deviation from Assumed Average (92)
90	-2
85	-7
90	-2
95	+3
Sum of deviations	(-8)
Average deviation	(-2)

Assumed average plus average deviation is true average.

$$92 \quad + \quad (-2) \quad = \quad 90$$

5. Have pupils use this method of finding the average.

Problem: Laura's marks on a series of five tests were 82, 72, 76, 84, 91. What is her average mark?

a. Have pupils study the marks and assume an average. Assume 80.

1) Set up table:

Test Marks	Deviations from Assumed Average (80)
82	+2
72	-8
76	-4
84	+4
91	+11

Sum of deviations +5

Average of deviations is +1

2) Then the true average is 80 + (1) or 81.

b. Have pupils check this average by assuming another average and computing. Assume 84.

1) Set up table:

Test Marks	Deviations from Assumed Average (84)
82	-2
72	-12
76	-8
84	0
91	7

Sum of deviations -15

Average of deviations is -3

Assumed average + average deviation = true average.

84 + (-3) = 81

2) Check with result in a.

6. If there is sufficient time and pupils are interested, the following proof that the assumed average + the average of the deviation equals the true average can be presented.

Let x represent the assumed average

d_1 the deviation of the first score from the average

d_2 the deviation of the second score from the average

d_3 the deviation of the third score from the average

d_4 the deviation of the fourth score from the average

.
.
.

d_n the deviation of the nth score from the average

Then

first score $= x + d_1$

second score $= x + d_2$

third score $= x + d_3$

fourth score $= x + d_4$

.
.
.

nth score $= x + d_n$

Adding

$$\text{Sum of scores} = nx + (d_1 + d_2 + d_3 + d_4 + \dots + d_n)$$

Dividing by number of scores

$$\frac{\text{Sum of scores}}{n} = \frac{nx + (d_1 + d_2 + d_3 + d_4 + \dots + d_n)}{n}$$

Simplifying

$$\frac{\text{Sum of scores}}{n} = x + \frac{(d_1 + d_2 + d_3 + d_4 + \dots + d_n)}{n}$$

Interpreting: The average of the scores equals the assumed average, x, plus the sum of the deviations divided by the number of scores.

6. Practice: Finding the averages in the following examples by the method of deviations:

   a. The boys on a team weigh 120 lbs., 134 lbs., 138 lbs., 115 lbs. and 117 lbs. What is their average weight?

   b. Three boys each measured a line. Their measures were 3.5 cm., 3.55 cm., 3.41 cm. What is the average of their measures?

c. The average barometric pressure on five successive days was: 30.23 in., 30.20 in., 29.95 in., 29.82 in. and 29.75 in. What was the average barometric pressure for the five days?

d. Four boys each measured the length of a room. Their measures were 12 ft. 6 inches, 12 ft. 5 inches, 12 ft. 7 inches, and 12 ft. $6\frac{1}{2}$ inches. Find the average of their measures.

e. The temperature at noon on four successive January days was:

8° , -5°, -3° , 4°. What is the average of the temperatures?

XII. Properties of Division

A. Suggested Procedure

1. Use the discovery method to elicit from pupils whether the following properties hold for the operation of division.

a. Commutative

Compare:

$2 \div 4 = \frac{1}{2}$

$4 \div 2 = 2$

$\frac{1}{2} \neq 2$. Therefore, the commutative property does not hold for division.

b. Associative

Compare:

$(8 \div 4) \div 2 = 1$

$8 \div (4 \div 2) = 8 \div 2 = 4$

$1 \neq 4$. Therefore, the associative property does not hold for division.

c. Distributive (division over addition)

1) $(8 \div 4) \div 2 = 12 \div 2 = 6$

$(8 \div 2) + (4 \div 2) = 4 + 2$ or 6

Then $(8 + 4) \div 2 = (8 \div 2) + (4 \div 2)$

2) Does $2 \div (8 + 4) = (2 \div 8) + (2 \div 4)$?

$$2 \div (8 + 4) = 2 \div 12 = \frac{1}{6}$$

$$(2 \div 8) + (2 \div 4) = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}$$

Then $2 \div (8 + 4) \neq (2 \div 8) + (2 \div 4)$

On the basis of this and several similar examples, pupils conclude that division may be distributed over addition, provided the addition is in the dividend. When the addition is in the dividend, the division may be expressed as a corresponding multiplication. The distributive property holds for multiplication.

B. Suggested Practice

1. Perform the indicated divisions.

Illustration:

$(-16) \div (-4) \div (+2)$

Since the associative principle does not hold for division, we must perform the divisions in order from left to right.

Then $(-16) \div (-4) \div (+2) = (+4) \div (+2) = +2$

a. $(+28) \div (-2) \div (+7)$

b. $(+\frac{1}{2}) \div (-\frac{3}{4}) \div (-\frac{1}{3})$

c. $(-1.5) \div (-.3) \div (+5)$

2. Use the distributive principle of division over addition to show how the following quotients may be obtained.

Illustration:

$\frac{+625}{+25} = +25$ because $\frac{(+600) + (+25)}{+25} = \frac{(+600)}{(+25)} + \frac{(+25)}{(+25)} = (+24) + (+1) = +25$

a. $\frac{-44}{+4} = -11$

b. $\frac{+435}{-5} = -87$

c. $\frac{-360}{-15} = +24$

d. $\frac{+6250}{+50} = +125$

CHAPTER III

THE USE OF LETTERS IN ALGEBRA

This chapter contains material which may be used by the teacher to reinforce and extend the pupils' understanding of the meaning of variable. In addition, teaching procedures are suggested to help the pupils develop an understanding of: the meaning of factors and exponents, how to evaluate formulas and algebraic expressions, and how to translate English phrases and sentences into algebraic expressions and open sentences.

I. The Letter as a Variable

A. Suggested Procedure

Review with pupils

1. Sets and set elements
2. Statements and open sentences
3. Meaning of variable, replacement set, domain
4. Meaning of solution set

B. Suggested Practice

1. A mathematical sentence, like an English sentence, expresses a complete thought. Tell whether each of the following is a sentence.

   a. $3 - 7 = -4$

   b. $2 + 9$

   c. $4 + 10 > 12$

   d. $5 + (-8)$

   e. $1 + 6 < 10$

2. Tell whether each of the following sentences is true, false, or open:

   a. There is one child in every family.

   b. $6 + 5 = 10$

   c. $8 + 3 > 10$

   d. His book is on the shelf.

   e. It is an even number.

f. $4 + 2 < 4 + 2$

g. This number is an odd number.

h. $\square - 10 = 5$

i. $2 + 4 < 3$

j. $\square + 1 = -5$

3. Which of the following are open sentences and which are statements? Explain.

a. It was a great football team.

b. He invented the cotton gin.

c. $x \cdot 4 = 12$

d. $5 + (-2) = 3$

e. $4 \times 3 > 5 \times 2$

f. All cats have two legs.

g. $4 + y = 9$

h. $x - 5 = 13$

i. $(+2) - (-1) = +3$

j. $13 - n > 10$

4. Circle the variable in each of the following open sentences:

a. He is a policeman.

b. It is made of carbon.

c. $3x = 12$

d. $n < 4$

e. $3 + a = 15$

f. She is wearing a red dress.

g. $y \times 5 = 25$

h. $\square + 2 = 7$

i. $\frac{1}{2}$ of $\triangle = 10$

j. $12 > n + 3$

5. Indicate an appropriate replacement set for each of the variables in the open sentences in 3.

Note: The solution set consists of all the numbers in the domain of the variable which will make the sentence true. For example, in 12 a + 5, the solution set is the set of numbers which when added to 5 will give a sum less than 12.

If the replacement set of a is $\{2, 3, 5, 7, 8, 9\}$, the solution set is $\{2, 3, 5\}$.

6. Indicate the solution set for each of the following inequalities where the replacement set of each variable is the set of all natural numbers:

   a. $14 > 6 + r$

   b. $n + 4 < 7$

   c. $8 + y < 10$

   d. $11 - m < 11$

   e. $5 + x > 8$

II. Using Variables in Formulas

A. Suggested Procedure

1. Have pupils recall that in the formula $p = 4s$, the letter p represents the number of units in the perimeter. The same letter p appears in the formula $i = prt$. Here again the letter p represents a number - the number of dollars in the principal. Pupils should be led to realize that in each case the letter represents a number drawn from an appropriate replacement set.

2. Have pupils consider the formulas

   $d = 2r$ $\qquad$ $d = rt$ $\qquad$ $i = prt$

   Reinforce the pupils' understanding that while r is used in each of the formulas with a different meaning, in each case r represents a number. In $d = 2r$, r represents a number from the set of all numbers which express the number of units of length of the radius.

   In $d = rt$, the domain of r is the set of all numbers which express the number of units of distance traveled in a unit of time.

   In $i = prt$, r represents a number from the set of all per cents.

3. Guide pupils to realize that a formula is an example of an open sentence. Thus, the formula

   $$d = 2r$$

   is an open sentence with two variables, and the formula

   $$d = rt$$

   is an open sentence with three variables.

   Whereas the formula $d = rt$ is a correct expression of a relationship among distance, rate and time, it becomes a true statement only when appropriate replacements are made for d, r, and t.

   If in the formula $d = rt$, we replace r by 40, t by 2, and d by 80, the resulting statement $80 = 40 \cdot 2$ is true.

B. Suggested Practice

1. Write an appropriate replacement set for the underlined variable in each of the following formulas:

   a. $p = 4\underline{s}$ Answer: $\{1\}$ or $\{1, 2\}$ etc.

   b. $A = \underline{l}w$

   c. $i = 36\underline{y}$

   d. $F = 1.8\underline{C} + 32$

   e. $c = 2\pi\underline{r}$

   f. $A = \frac{1}{2}\underline{b}h$

   g. $c = np$ (cost = number x price)

   What is the largest possible replacement set for each of the above? (The largest replacement set in example a is {all positive numbers}.)

2. Choose a replacement for each variable which will make the formula a true statement.

	Replacement Sets	
a. $p = 4s$	p: $\{6, 7, 8\}$	s: $\{1, 2, 3\}$
b. $A = \ell w$	A: $\{5, 6, 7, 8\}$ ℓ: $\{1, 2, 3\}$	w: $\{1, 2\}$
c. $i = 36y$	i: $\{72, 108\}$	y: $\{1, 2\}$
d. $d = 2r$	d: $\{4, 5, 6\}$	r: $\{1\frac{1}{2}, 2, 2\frac{1}{2}, 3\frac{1}{2}\}$
e. $F = 1.8C + 32$	F: $\{23, 82, 164\}$	C: $\{-5, 10, 20\}$

III. Factors and Exponents

A. Suggested Procedure

1. Have pupils use symbols to indicate the product of the following:

two and three	2 x 3
five and two	5 x 2
seven and one	7 x 1
three and three	3 x 3

Have them recall that product implies multiplication and that 2 and 3, for example, are called factors of the product. The number 6 also represents the product of two and three.

2. Have pupils note that 3 x 3 or 9 represents a product in which the same factor has been used twice. They already know that 3 x 3 can be written as 3^2 which is read as "3 squared," or "3 square."

Have pupils read the following:

5×5 or $5^2 = 25$ $\qquad$ 7×7 or $7^2 = 49$

Have pupils express both 5^2 and 2 x 5 in another form and compare the values obtained. ($5^2 = 5 \cdot 5$ and $2 \times 5 = 5 + 5$)

Elicit from pupils that 5^2 means 5 is taken as a factor twice. To avoid a common error, do not permit pupils to say, "2 times 5" or "5 is multiplied by itself twice."

Have pupils note that 2 x 2 x 2 or 8 represents a product in which the same factor has been used three times. The product 2 x 2 x 2 can be written as 2^3 which is read as "2 cubed" or "2 cube."

Have pupils read the following:

$4 \times 4 \times 4$ or $4^3 = 64$ $\qquad$ $6 \times 6 \times 6$ or $6^3 = 216$

Have pupils express both 4^3 and 3 x 4 in another form and compare the values obtained. Have them do the same for 6^3 and 3 x 6.

After several similar examples, have the pupils consider the familiar formulas:

$$A = s^2 \qquad V = e^3$$

Elicit that s^2 means s is used as a factor twice.

Have pupils compare s^2 and 2s when s is replaced by 5.

Elicit that e^3 means e is used as a factor three times. Have pupils compare e^3 and 3e when e is replaced by 2.

In the following examples, have the pupil state how many times y is used as a factor:

y^2 $\qquad$ y^4 $\qquad$ y^7

3. Lead the pupil to see that in an expression of the form y^7, we need some way of describing the numbers involved. The y, which indicates the number we are going to use as a factor several times, is called the base; the 7, which indicates how many times the factor y is used, is called the exponent. Thus, y^7 means a number consisting of seven equal factors y.

A number which can be expressed by means of a base and exponent is called a power. The exponent 1, which is rarely written, means the base is used just once. Then y^1, the first power of y, is the same as y.

Have pupils practice reading the following expressions:

y^4 "the fourth power of y" also, "y to the fourth power" or "y to the fourth."

y^7 5^4 c^6

4. Guide pupils to see that in an expression such as $5 \cdot 3^2$, the 5 is the exponent of the base 3. Its value is $5 \cdot 9$ or 45. In an expression such as $(5 \cdot 3)^2$, the 2 is the exponent of the base $5 \cdot 3$, because the expression has been enclosed in a symbol of grouping. Its value is 15^2 or 225.

   Have pupils compare the following:

   $4 \cdot 2^3$ and $(4 \cdot 2)^3$

   $5 \cdot 6^2$ and $(5 \cdot 6)^2$

   Have them compute the value of each of the above.

B. Suggested Practice

1. Express each of the following as the product of two or more factors:

   18 14 5 36 x^2 y^3

2. Rewrite each of the following in a shorter form:

   a. $y \cdot y$

   b. $x \cdot x \cdot x$

   c. 4 cubed

   d. 13 squared

   e. S used as a factor 5 times

   f. p used as a factor 8 times

   g. $2 \cdot a \cdot a$

   h. $10 \cdot b \cdot b \cdot b$

   i. the sixth power of x

   j. $r \cdot r$

3. Fill in the columns for each of the following expressions:

Power	Base	Exponent	Meaning	Value (where possible)
2^4	2	4	$2 \cdot 2 \cdot 2 \cdot 2$	
3^3				
$(\frac{1}{2})^2$				
$(.4)^2$				
x^5				
m^6				

4. Have the pupil find the value of each of the following:

2^6

6^2

5^4

10^3

$(\frac{1}{2})^3$

$(.5)^2$

4×3^2

$(5 \cdot 2)^2$

5. Express in another form:

   a. 100^2 (Elicit answer: 100×100)

   b. 2×100 (Elicit answer: $100 + 100$)

   c. a^3 $(a \cdot a \cdot a)$

   d. $3a$ $(a + a + a)$

6. State whether each of the following is true or false, or cannot be determined.

   $(.2)^2 = .4$ $\qquad$ $3^2 = 3 \cdot 3$ $\qquad$ $m^3 = 3m$

IV. Evaluating Formulas and Algebraic Expressions

A. Suggested Procedure

1. Have pupils review the evaluation of simple formulas previously learned.

   a. Find the value of d in the formula for the diameter of a circle, $d = 2r$, when the variable r is replaced by the element 12 from its domain.

   b. Using the formula $p = 4s$, find the value of p, corresponding to each value of s, if the domain of s is $\{2, 7, 10\}$.

   c. Using the formula $A = \ell w$, find A when $\ell = 6$ and $w = 5$.

   d. Using the formula $i = prt$, find i when $p = 200$, $r = .04$, and $t = 2$.

2. Have the pupils review that when several operations are involved in evaluating a number expression, we follow an agreed-upon order of operation:

   Multiplication and division are to be performed in order, from left to right, before addition and subtraction.

   Note: See Chapter I, Section III, pages 4-6.

   a. $6 + 2\cdot5$ means $6 + 10$ or 16 and not $8\cdot5$ or 40

      Ask the pupils how they would write $6 + 2\cdot5$ if they wanted the answer 40. They will suggest using a symbol of grouping such as $(6 + 2)5$.

   b. The pupils should also be reminded that operations in parentheses are to be evaluated first. For example,

      $2(5 + 3) = 2\cdot8 = 16$ $\qquad$ $(4 + 3)^2 = 7^2 = 49$

   c. Have pupils do several examples such as:

      1) $3 + 7\cdot4$
      2) $(3 + 7)\cdot4$
      3) $3\cdot7 + 4$
      4) $5\cdot6 + 4\cdot3$
      5) $\frac{12}{3} \div 4$
      6) $12 \div 2 \times 3 - 1$

3. Have pupils consider the formula $p = 21 + 2w$.

   Elicit from the pupils that if we replace the variable 1 by 8, and the variable w by 4, we obtain the following:

$$p = 2\cdot8 + 2\cdot4$$

$$\therefore p = 16 + 8$$

$$\therefore p = 24$$

4. Tell pupils that the process of finding the value of an expression for particular values of the variables is called evaluation.

5. Have pupils evaluate the following expressions by replacing the variables as indicated.

a. $2l + 2w$ where $l = 10$ and $w = 8$

b. $3x - 2$ where the domain of x is $\{3, 6, 12\}$ (Find value of expression for each element in domain.)

c. $mn + n$ where $m = 7$ and $n = 3$

d. $\frac{a}{b} + b$ where $a = 10$ and $b = 2$

6. Have pupils evaluate the formulas:

$A = s^2$ where $s = 3$

$V = e^3$ where $e = 4$

7. Have them consider the formula $s = 16t^2$.

Remind them that t is the base to which the exponent 2 belongs, so that t is the factor that is taken twice, and not 16t.

If we wished to express 16t as a factor twice, we would write it as $(16t)^2$. Therefore, in evaluating the formula for $t = 3$, we would write

$$s = 16t^2$$

$$s = 16 \cdot 3^2$$

$$s = 144$$

8. In a similar way, have the pupils evaluate the following expressions:

a. $3a$ when $a = 10$

b. $3a^2$ when $a = -4$

c. $x + 4$ when $x = 12$; $x = 4$

d. $(\frac{1}{y})(x + 4)$ when $x = 16$, $y = 4$

e. x^2y^3 when $x = 3$ and $y = 2$

f. πr^2 when $\pi \approx \frac{22}{7}$ and $r = 7$

g. $a \div b + y$ when $a=6$, $b=3$, $y=5$

h. m^2n^2 when $m = 2$, $n = 4$

i. $(ab)^3$ when $a = 3$, $b = 2$

j. $3(a + b)^2$ when $a = 2$, $b = -3$

k. $(a + b)^2$ when $a = 4$, $b = 1$

l. $a^2 + b^2$ when $a = 4$, $b = 1$

B. Suggested Practice

1. **Evaluate:**

   a. Find the values of A in $A = s^2$ if the domain of s is $\{6, 15, 20\}$.

   b. Find the value of A in $A = \pi r^2$ if $\pi \approx \frac{22}{7}$ and r is 14.

   c. Find the values of V in $V = e^3$ if the domain of e is $\{5, \frac{1}{2}\}$.

   d. Find the values of $5x + 6$ if the domain of x is $\{-7, 9, 12\}$.

   e. Find the values of $xy + y$ for the following values of x and y:

x	y	xy + y
-4	7	
8	-6	
10	12	

   f. Find I in $I = \frac{E}{R}$ if $E = 2.7$ and $r = 3$.

   g. Find A in $A = P + Prt$ if $P = 200$, $r = .04$, $t = 2\frac{1}{2}$

   h. Find s in $s = \frac{1}{2} at^2$ if $a = -32$ and $t = 4$.

   i. Find the value of $7x - 4$ and of $7(x - 4)$ when $x = 5$.

2. Solve the following problems by making the proper substitutions in the correct formulas.

   a. How much fencing is needed to build a fence around a lot $47\frac{1}{2}$ feet wide and 123 feet long?

   b. Find the area of a baseball diamond which is a square 90 feet on each side.

   c. A piece of luggage is 24 inches long, 16 inches wide and 8 inches deep. Find the volume of the luggage.

   d. Which is larger, a circular rug with a radius of 3 feet, or a rectangular rug 6 feet by $4\frac{1}{2}$ feet?

   e. How many gallons of oil will a cylindrical oil tank hold if it is 20 feet long and 7 feet in diameter?

V. Translation of English Phrases and Sentences Into Algebraic Expressions and Open Sentences

A. Suggested Procedure

1. Have pupils review distinction between a sentence and a phrase.

The following are sentences because each expresses a complete thought:

"He has brown hair."   "$12 + 9 = 21$"

whereas the following are phrases:

"Three coins"   "$2 + 4$"

2. Review the translation of English sentences and phrases in which variables were not involved. (See Chapter I, Section IV, pages 7,8)

3. Have pupils translate the following:

   a. seven more than five   translated as $5 + 7$

   Note: Have pupils observe the difference between the phrase "seven more than five" ($5 + 7$) and the sentence "Seven is more than five." ($7 > 5$)

   two more than eight   translated as $8 + 2$

   three more than fifty   translated as $50 + 3$

   Have pupils use a variable, such as n, to translate:

   two more than any number   translated as $n + 2$

   b. two increased by five   translated as $2 + 5$

   six increased by three   translated as $6 + 3$

   any number increased by two   translated as $n + 2$

4. In a similar manner, develop decreased by, less than, twice a number, sum, difference, product, etc.

5. Emphasize that addition and multiplication are commutative operations, but subtraction and division are not. An expression such as "6 added to 7" written $7 + 6$ gives the same result as $6 + 7$ because addition is commutative. An expression such as "the product of 6 and 7" written as $6 \cdot 7$ gives the same result as $7 \cdot 6$ because multiplication is commutative.

   However, an expression such as "6 less than 7" written $7 - 6$ is not the same as "7 less than 6" which is written $6 - 7$. Subtraction is not commutative. An expression such as "12 divided by 2", written $\frac{12}{2}$ is not the same as "2 divided by 12" which is written $\frac{2}{12}$, because division is not commutative.

6. Have pupils learn to translate expressions using a grouping symbol ( ) when necessary.

a. twice the sum of 4 and 5	$2(4 + 5)$
b. fifteen decreased by the sum of 6 and 9	$15 - (6 + 9)$
c. twice a number, increased by the difference of the number and 6	$2n + (n - 6)$
d. half the sum of a number and 4, decreased by the difference of the number and 1	$\frac{1}{2}(n + 4) - (n - 1)$

7. Have pupils use y as the variable and translate open sentences expressed in English into equations.

   a. A number increased by 18 is 36.

   b. Twice a number, decreased by 6 is 14.

   c. A number increased by three times the sum of 3 and 5 is 30.

   d. The sum of three fourths of a number and 7 is 27.

   e. Four increased by a number is twice the difference of 8 and 3.

B. Suggested Practice

1. Use x for the variable and write each of the following in mathematical symbols. Tell which are sentences and which are phrases.

   a. The sum of 4 and a number

   b. A number decreased by 4 equals 2

   c. 6 is less than a number

   d. 6 less than a number

   e. 12 divided by a number

   f. 5 multiplied by a number is 10

2. Translate the following verbal expressions into algebraic language:

   if n represents a certain number, express in terms of n

   a. 5 times the number
   b. the number increased by 4
   c. the number diminished by 8
   d. 1.5 divided by the number
   e. six less than twice the number
   f. the product of 5 and the number
   g. 7 added to the product of 3 times the number
   h. 4 times the sum of the number and 3

3. Express each of the following in symbols:

   a. 20 divided by 4 and the result added to 15

   b. the result obtained when the sum of 7 and 8 is divided by 4

4. Express in terms of n

   a. twice a number added to 15

   b. 18 less than 4 times a number

5. If g represents the larger of two numbers and s represents the smaller, express in terms of g and s:

   a. 5 times the larger increased by twice the smaller

   b. the sum of both numbers multiplied by 5

   c. the square of the sum of the larger and the smaller

6. Exercises similar to V-A-7.

CHAPTER IV

USING VARIABLES TO EXPRESS NUMBER PROPERTIES

In this chapter the teacher will find suggested procedures for helping pupils develop understanding and skill in: using variables to express number properties, using the distributive property to simplify expressions and to solve simple linear equations, using equations to solve verbal problems.

I. Properties of Addition and Multiplication

A. Suggested Procedure

1. Using variables to express the commutative property of addition

   a. Review commutative property of addition.

   Have pupils study the following and recall the property that is involved:

   $8 + 5 = 5 + 8$ $4 + (-3) = (-3) + 4$ $\frac{1}{2} + \frac{1}{6} = \frac{1}{6} + \frac{1}{2}$

   b. Ask pupils how we could express the property illustrated by the above equations by means of a single equation.

   Have them recall that a letter is used to represent some unspecified number drawn from a set.

   Pupils may suggest the sentence $x + y = y + x$, where x and y represent any two numbers.

   Have them consider whether this expresses the commutative property of addition by trying various replacements for x and y.

   If $x = 5$ and $y = 2$, then $5 + 2 = 2 + 5$ true

   If $x = -3$ and $y = -8$, then $(-3) + (-8) = (-8) + (-3)$ true

   If $x = \frac{3}{4}$ and $y = -\frac{1}{2}$, then $\frac{3}{4} + (-\frac{1}{2}) = (-\frac{1}{2}) + \frac{3}{4}$ true

   c. Have pupils attempt to find a counter-example (one that will disprove it).

   Note: Point out to pupils that although one counter-example may disprove a statement, the present lack of such a counter-example does not prove a statement because a counter-example may be found at a later date.

d. Have pupils realize that therefore it is reasonable to assume that

$x + y = y + x$ is true for all numbers x and y.

Have them see that variables have been used to state the commutative property of addition.

2. Using variables to express the commutative property of multiplication

a. Review commutative property of multiplication.

b. In a similar fashion, develop with pupils a generalization of the commutative property of multiplication.

Have them realize that it is reasonable to assume that $xy = yx$ is true for all numbers x and y, and that variables have been used to state the commutative property of multiplication.

3. Using variables to express the associative property of addition

a. Review associative property of addition.

Have pupils study the following and recall the number property that is involved:

$(4 + 7) + 8 = 4 + (7 + 8)$ $(-3 + 5) + 8 = -3 + (5 + 8)$

b. Have pupils suggest the following symbolic statement of the property:

$(x + y) + z = x + (y + z)$ where x and y and z represent any numbers.

Have them consider whether this expresses the associative property of addition.

Have them try various replacements and verify that true statements result.

If $x = 2$, $y = 3$, $z = 4$, then $(2 + 3) + 4 = 2 + (3 + 4)$ true

If $x = -8$, $y = 5$, $z = 3$, then $(-8 + 5) + 3 = -8 + (5 + 3)$ true

If $x = \frac{3}{8}$, $y = \frac{1}{2}$, $z = \frac{1}{4}$, then $(\frac{3}{8}+\frac{1}{2}) + \frac{1}{4} = \frac{3}{8}+(\frac{1}{2}+\frac{1}{4})$ true

c. Have pupils attempt to find a counter-example.

d. Have pupils realize that it is reasonable to assume that for all numbers, x, y, z, $(x + y) + z = x + (y + z)$ is true.

4. Using variables to express the associative property of multiplication

a. Review associative property of multiplication.

b. In a similar fashion, develop with pupils a generalization of the associative property for multiplication.

Have them conclude that for all numbers x, y, z, $(xy)z = x(yz)$.

5. Using variables to express the distributive principle of multiplication over addition

a. Review the distributive property of multiplication over addition.

Have pupils recall the linking together of the operations of multiplication and addition by observing the pattern in the following:

$6\cdot(4 + 3) = 6\cdot4 + 6\cdot3$

$-2(9+6) = (-2)(9) + (-2)(6)$

$\frac{1}{2}(-4 + 8) = \frac{1}{2}(-4) + \frac{1}{2}(8)$

b. Have pupils suggest the following symbolic statement of the property:

$x(y + z) = xy + xz$ where x, y, and z represent any numbers.

Have them consider whether this expresses the distributive property of multiplication over addition by trying various replacements for x, y and z.

	Statement
when $x = 17$, $y = 5$, $z = 6$, then $17\cdot(5 + 6) = 17\cdot5 + 17\cdot6$	true
when $x = -4$, $y = 8$, $z = 3$, then $-4(8 + 3) = (-4\cdot8) + (-4\cdot3)$	true
when $x = 14$, $y = \frac{1}{7}$, $z = \frac{3}{14}$, then $14(\frac{1}{7} + \frac{3}{14}) = 14\cdot\frac{1}{7} + 14\cdot\frac{3}{14}$	true

c. Have pupils attempt to find a counter-example.

d. Have them realize that it is reasonable to assume that for all numbers x, y, and z

$x\ (y + z) = xy + xz$ is true

e. Have pupils realize that since multiplication is commutative, other forms of the distributive property can be derived:

$(y + z)x = xy + xz$ for all values of x, y and z.

$(y + z)x = yx + zx$ for all values of x, y and z.

B. Suggested Practice

1. Use the commutative principle to write another name for each sum and product.

Illustration: $2 + 5$ Solution: $5 + 2$

a. $4 + y$

b. $z + 9$

c. $a + b$

d. $3 \cdot 2$

e. $6 \cdot 6^2$

f. $2^2 \cdot 3^2$

g. $3 \cdot 5$

h. $12 + 7$

i. $a \cdot b$

2. State the commutative principle for addition, and for multiplication in terms of x and y.

a. Illustrate the properties with these numbers:

$x = 15$ and $y = 7$

$x = 35$ and $y = 29$

$x = 4.1$ and $y = 9.3$

$x = -10$ and $y = 18$

b. Each of the following equations is true for all possible replacements of any indicated variables, because of the commutative or associative principle, or both.

Answer each part by writing "A" for associative, "C" for commutative and both "C" and "A" for both commutative and associative principles.

$6 + (9 + 2) = (6 + 9) + 2$ Solution: A

$4 + (x + y) = (4 + x) + y$

$4 + (x + y) = 4 + (y + x)$

$(a + b) + c = (b + a) + c$

$4 + (x + y) = (x + 4) + y$

$(9 + a) + b = a + (9 + b)$

3. Use the commutative and associative principles to simplify the following expressions:

a. $3(2x)$ Solution: $3(2x) = (3 \cdot 2)x$ or $6x$

b. $4(3y)$

c. $-\frac{1}{2}(2x)$

d. $.8(.2b)$

e. $3n(4)$ Solution: $3n(4) = 3(n \cdot 4) = 3(4 \cdot n) = (3 \cdot 4)n = 12n$

f. $8z(6)$

g. $\frac{2}{3}y(4)$

h. $-15m(\frac{4}{3})$

i. $3a(4b) = 3(a \cdot 4)b$ associative

$= 3(4a)b$ commutative

$= (3 \cdot 4)(ab)$ associative

$= 12ab$

j. $(-2m)(-6n)$

k. $(4p)(-9q)$

l. $(2ab)(4bc)(8ac)$

Solution: By the commutative and associative properties of multiplication, we may arrange the factors in this expression in any order we wish without changing its value. Then we may write:

$2 \cdot 4 \cdot 8 \cdot a \cdot a \cdot b \cdot b \cdot c \cdot c$ as an expression equivalent to the above.

By using superscript notation (exponents), we then write this as:

$64a^2b^2c^2$

4. Which of the following are illustrations of the distributive property?

a. $x(4 + 23) = x(4) + x(23)$ c. $3(x + y) = 3x + y$

b. $250y + 75y = (250 + 75)y$ d. $(500 + 9)b = 500b + 9b$

5. Fill in the blanks so that each statement is an illustration of the distributive property:

a. $6(8 + 9) =$ ____ Solution: $6 \times 8 + 6 \times 9$

b. ____ $= 7 \cdot 2 + 7 \cdot 6$

c. $4\cdot(\_\_\_\_ + \_\_\_\_) = \_\_\_\_\cdot 5 + \_\_\_\_\cdot 6$

d. $2(a + b) = \_\_\_\_$

e. $2(15 + 10x) = \_\_\_\_$

f. $12x + 35x = \_\_\_\_$

g. $\_\_\_\_ = xy + \frac{1}{2}y$

h. $9\cdot 2x + 9\cdot 2y = \_\_\_\_$

II. Identity Elements and Inverse Elements

A. Suggested Procedure

1. Help pupils reach a generalization concerning the additive identity (additive property of zero).

   a. Have pupils review that when zero is added to a number, the sum is the number itself, e.g.,

   $5 + 0 = 5 \qquad (-2) + 0 = -2 \qquad \frac{22}{7} + 0 = \frac{22}{7} \qquad .8 + 0 = .8$

   b. The number to which zero is added keeps its identity under this addition. Therefore, zero is called the identity element of addition or the additive identity.

   c. Have pupils suggest the following symbolic statement concerning the additive identity:

   $x + 0 = x$ or $0 + x = x$

   By trying various replacements for x, pupils see that these appear to be true statements for every possible replacement of the variable.

2. Help pupils reach a generalization concerning the multiplicative identity (multiplicative property of one).

   a. Have pupils recall that there is a number which preserves the identity of every other number under multiplication. Have them recall that the product of one and any other number is the other number, e.g.,

   $1 \times 3 = 3 \qquad 8 \times 1 = 8 \qquad (-9) \times 1 = -9 \qquad 1 \times \frac{1}{2} = \frac{1}{2}$

b. The number by which 1 is multiplied keeps its identity. Therefore, 1 is called the identity element of multiplication, or the multiplicative identity.

c. Have pupils suggest the following symbolic statement concerning the multiplicative identity:

$1 \cdot x = x$ or $x \cdot 1 = x$

By trying various replacements for x, pupils see that these appear to be true statements for every possible replacement of the variable.

3. Help pupils reach a generalization concerning the additive inverse (opposite).

a. Have pupils consider

$1 + (-1) = 0$ $\quad$ $(-1) + (+1) = 0$

$2 + (-2) = 0$ $\quad$ $(-2) + (2) = 0$

$3 + (-3) = 0$ $\quad$ $(-3) + (3) = 0$

b. Have pupils recall that the sum of a number and its opposite is zero.

c. Have pupils suggest the following symbolic statement concerning the additive inverse:

$x + (-x) = 0$ For which values of x does this hold?

d. By trying many possible replacements, have them realize that this is a true statement for every possible replacement of the variable. (positive, negative, or zero)

e. Tell pupils that if the sum of two numbers is zero, they are called additive inverses. Each is the opposite of the other.

4. Help pupils reach a generalization concerning the multiplicative inverse (reciprocal).

a. Have pupils recall that a number multiplied by one is the number itself.

b. Have them consider whether there is a number, which when multiplied by another will yield a product of one. For example, given the number 6, is there some number N such that

$N \times 6 = 1$?

c. Have them realize that $\frac{1}{6} \times 6 = 1$.

d. Tell pupils that $\frac{1}{6}$ is called the reciprocal or multiplicative inverse of 6; 6 is also the multiplicative inverse of $\frac{1}{6}$.

e. Tell pupils that two numbers whose product is one (1) are called multiplicative inverses. Each number is the multiplicative inverse of the other.

f. Have pupils state the multiplicative inverse of each of several numbers, e.g., have them tell by what number each of the following is multiplied to give a product of one.

$\frac{1}{3}$ $\quad \frac{1}{2}$ $\quad -\frac{1}{8}$ $\quad -\frac{3}{4}$ $\quad \frac{7}{10}$ $\quad -\frac{4}{5}$ $\quad -6$

g. Have pupils consider whether zero has a multiplicative inverse.

If zero has a multiplicative inverse, we could represent it by N. Then N x 0 must equal 1. Can we find a replacement for N which will make this statement true?

1) Have pupils consider how 0 acts under multiplication.

Have them recall that multiplication involving zero results in a product of zero, e.g.,

$2 \times 0 = 0 \quad 0 \times (-10) = 0 \quad \frac{2}{3} \times 0 = 0 \quad 1 \times 0 = 0$

2) Have them suggest the following symbolic statement concerning multiplication involving zero.

$0 \cdot N = 0$ or $N \cdot 0 = 0$

By trying various replacements for N, pupils see that these appear to be true statements for every possible replacement of the variable. This property of zero is called the multiplicative property of zero.

3) Pupils should realize that since the open sentence $N \cdot 0 = 1$ can never be true for any replacement of N by a number, zero does not have a multiplicative inverse.

h. Have pupils suggest the following symbolic statement concerning the multiplicative inverse: $\frac{1}{x} \cdot x = 1$

Have them realize, by trying different replacements, that this is a true statement for every replacement of the variable x by a number except the number zero (0).

Note: Since $N \cdot 0 = 0$ for any value of N, we must exclude division by zero.

OPTIONAL - Proofs of Some Basic Number Properties

On the basis of several properties concerning numbers, which the pupils have accepted, they may be guided to see that other truths follow from these properties by logical reasoning. The process of reasoning from some accepted properties to other truths is called proof.

5. Help pupils prove the multiplicative property of 0. That is, prove $x \cdot 0 = 0$.

We would proceed as follows:

$x(1 + 0) = x \cdot 1$ $(1 + 0 = 1)$

$x(1 + 0) = x \cdot 1 + x \cdot 0$ (Distributive property)

$x \cdot 1 + x \cdot 0 = x \cdot 1$ (Both are equal to $x(1 + 0)$)

$x + x \cdot 0 = x$ (Multiplicative property of 1)

Since 0 is the additive identity, that is, the only number which when added to another number preserves the other number's identity, then $x \cdot 0 = 0$.

6. Help pupils prove that the additive inverse of 1 multiplied by x is the additive inverse of x. That is, prove $(-1)x = -x$.

We would proceed as follows:

$x[1 + (-1)] = 0$ (Multiplicative property of 0)

$x[1 + (-1)] = x(1) + x(-1)$ (Distributive property)

$= 1(x) + (-1)x$ (Commutative property)

$= x + (-1)x = 0$ (Multiplicative property of 1)

Since $x + (-1)x = 0$ and we assume each number x has one and only one additive inverse $(-x)$, then $-1(x) = -x$.

This is often referred to as the property of -1 Multiplication .

7. Help pupils prove $(-x)y = -(xy)$.

We proceed as follows:

$(-x)y = (-1 \cdot x)y$ (Property of -1 Multiplication)

$= (-1)(xy)$ (Associative property)

$= -xy$ (Property of -1 Multiplication)

Thus $(-x)y = -xy$

8. Help pupils prove $(-x)(-y) = xy$.

We proceed as follows:

$-x[y + (-y)] = 0$ (Multiplicative property of 0)

$-x[y + (-y)] = (-x)y + (-x)(-y)$ (Distributive property)

$= -xy + (-x)(-y) = 0$ (Property of -1 Multiplication)

Since $-xy + (-x)(-y) = 0$ and we assume each number has one and only one additive inverse, then $(-x)(-y) = xy$.

Guide pupils to see that the rule for multiplying two negative numbers is a special case of the above generalization, when x and y are restricted to positive numbers.

B. Suggested Practice

1. What is the additive inverse of each of the following?

a. -3 b. $4\frac{1}{2}$ c. 0 d. $-.6$

2. Complete by writing x or -x.

a. The additive inverse of x is _____.

b. The expression _____ + x = 0.

c. The additive inverse of -x is _____.

3. What is the multiplicative inverse of each of the following?

a. 3 b. $\frac{3}{4}$ c. -3 d. 1

4. Name the property of numbers which justifies each step in the following:

$$\begin{aligned}6 \times (3 \times \tfrac{1}{6}) &= 6 \times (\tfrac{1}{6} \times 3)\\ &= (6 \times \tfrac{1}{6}) \times 3\\ &= 1 \times 3\\ &= 3\end{aligned}$$

$$\begin{aligned}-5x(4 \times -\tfrac{1}{5}) &= -5x\ (-\tfrac{1}{5} \times 4)\\ &= (-5 \times -\tfrac{1}{5}) \times 4\\ &= 1 \times 4\\ &= 4\end{aligned}$$

III. Distributive Property of Multiplication Over Subtraction

A. Suggested Procedure

1. Have pupils consider the following:

   Does $3(7-4) = 3 \cdot 7 - 3 \cdot 4$?

   $3(7-4) = 3 \cdot 3$, or 9

   $3 \cdot 7 - 3 \cdot 4 = 21-12$, or 9

   Thus $3(7 - 4) = 3 \cdot 7 - 3 \cdot 4$.

2. After several such illustrations, have pupils realize that multiplication appears to be distributive over subtraction.

3. Help pupils reach a generalization concerning the distributive property of multiplication over subtraction.

   a. Have pupils suggest a symbolic statement concerning the distributive property of multiplication over subtraction:

   $x(y - z) = xy - xz$

   b. After trying many replacements, they should realize that this is a true statement for every possible replacement of the variables.

   Note: The proof of the symbolic statement in 3a will be found as an optional exercise in the Suggested Practice which follows.

B. Suggested Practice

1. Which of the following are illustrations of the distributive property?

   a. $(100 - 10)a = 100a - 10a$

   b. $4(20 - b) = 4(20) - (b)$

   c. $16 - 5x = (16 - 5)x$

   d. $(1.5 - c)1.3 = (1.3)(1.5) - (1.3)c$

   e. $16y - y = (16 - 1)y$

2. Fill in the blanks so that each statement is an illustration of the distributive property.

   a. $5(3 - 2) =$ _____.

   b. _____ $= 9.8 - 9.2$

   c. $a(6 - 4) =$ _____.

   d. $9x - 9y = 9$ (_____).

OPTIONAL

3. Prove $x(y - z) = xy - xz$

Solution:

$x(y - z) = x[y + (-z)]$ (Definition of subtraction)

$= xy + x(-z)$ (Distributive property)

$= xy + x[(-1)z]$ (Property of -1 Multiplication)

$= xy + x[z(-1)]$ (Commutative property)

$= xy + (xz)(-1)$ (Associative property)

$= xy + [-(xz)]$ (Property of -1 Multiplication)

$= xy - xz$ (Definition of subtraction)

IV. The Distributive Principle of Division Over Addition; Over Subtraction

A. Suggested Procedure

1. Have pupils consider the example

$$\frac{2}{6} + \frac{1}{6} = \frac{2 + 1}{6} = \frac{1}{2}$$

2. Have them recall that division by 6 is equivalent to multiplication by $\frac{1}{6}$ and therefore, $\frac{2 + 1}{6}$ may be expressed as $\frac{1}{6}\cdot(2+1)$.

Thus, $\frac{1}{6}(2+1) = \frac{1}{6}\ (2) + \frac{1}{6}(1)$ or $\frac{2}{6} + \frac{1}{6}$

Thus, division seems to be distributive over addition provided the addition is in the dividend.

In the same way, $\frac{7}{8} - \frac{3}{8} = \frac{7 - 3}{8} = \frac{1}{8}(7 - 3)$

Thus, $\frac{1}{8}(7 - 3) = \frac{1}{8}(7) - \frac{1}{8}(3)$ or $\frac{7}{8} - \frac{3}{8}$

Thus, division seems to be distributive over subtraction, provided the subtraction is in the dividend.

3. Help pupils reach a generalization concerning the distributive property of division over addition; over subtraction.

Have pupils suggest a symbolic statement concerning the distributive property of division.

$\frac{x + y}{z} = \frac{x}{z} + \frac{y}{z}$ $\frac{x - y}{z} = \frac{x}{z} - \frac{y}{z}$

4. After trying many replacements, they should realize that these are true statements for all values of x, y, z except z = 0.

5. Since subtraction can be expressed as addition, and since division can be expressed as multiplication, distribution of division over addition or subtraction can be justified by the distributive principle of multiplication over addition.

B. Suggested Practice

1. Which of the following illustrate a correct use of the distributive property?

a. $\frac{a}{4} + \frac{b}{4} = \frac{a + b}{4}$

b. $\frac{x + y}{5} = \frac{x}{5} + \frac{y}{5}$

c. $\frac{x}{6} - \frac{y}{6} = \frac{x - y}{6}$

d. $\frac{a - b}{x} = \frac{a}{x} - b$

e. $\frac{xy}{4} = \frac{x}{4} \cdot \frac{y}{4}$

Note: It should be pointed out to the pupil that division is not distributive over multiplication. Although the open sentence in example 1-e is true when x = 0 or y = 0, it is not true in general.

2. Fill in the blanks so that each statement is an illustration of the distributive property.

a. $\frac{x + 4}{3} = \frac{x}{3} + \frac{\_\_\_\_}{3}$

b. $\frac{\_\_\_\_}{2} = \frac{a}{2} - \frac{b}{2}$

c. $\frac{1}{2}(x + y) = \frac{\_\_\_\_}{2} + \frac{\_\_\_\_}{2}$

d. $\frac{a}{x} + \frac{b}{x} = \frac{a + b}{\_\_\_\_}$

V. Using the Distributive Property to Simplify Expressions

A. Suggested Procedure

1. Have pupils consider how the distributive property could be used to facilitate the following computation:

18 x 7 + 12 x 7 = (18 + 12)7 = 30 x 7 or 210

12 x 3 - 8 x 3 = (12 - 8)3 = 4 x 3 or 12

2. Have pupils consider the algebraic expression $3a + 5a$

   a. Have them realize that by the distributive property the sentence

      $3a + 5a = (3 + 5)a$ is true for every number a

   b. Therefore, $3a + 5a$ may be replaced by the shorter expression $(3 + 5)a$ or $8a$. This is a simplification of $3a + 5a$.

3. Have pupils consider the algebraic expression: $(2x + 8) + 3x$.

   Have pupils understand the simplification of the expression as follows:

$(2x + 8) + 3x = 2x + (8 + 3x)$	Associative principle
$= 2x + (3x + 8)$	Commutative principle
$= (2x + 3x) + 8$	Associative principle
$= (2 + 3)x + 8$	Distributive principle
$= 5x + 8$	Arithmetic fact

B. Suggested Practice

1. Which of the following open sentences are illustrations of the distributive property?

   a. $x(3 + 4) = 3x + 4x$

   b. $10a - 10b = 10(a - b)$

   c. $2(x + y) = 2x + y$ Since the multiplication by 2 has not been distributed, this is not an illustration.

   d. $2x(x + y) = 2x^2 + xy$

   e. $3ab - b^2 + (3a - b)b$

   f. $5xy + 10yz = 5y(x + 2z)$

   g. $(2a + 5)b = 2ab + 5 + b$

   h. $7x(y + z) = 7xy + z$

   i. $6n - mn = (6 - m)n$

2. Apply the distributive property and write each of the following in a form without parentheses:

a. $3(a + b)$

b. $(2 - 4x)y$

c. $(\frac{1}{2} + y)\frac{2}{3}y$

d. $.5m(9 + 1.2n)$

e. $-2(x + y)$

f. $-3(2\frac{1}{3}x - y)$

3. Apply the distributive property and express each of the following as products:

a. $3y + 2y$

b. $5r - 2r$

c. $2xy + 2x$

d. $4bc - 2bc$

e. $5(x + y) + 2(x + y)$

4. Use the associative, commutative and distributive properties to write the following expressions in simpler form:

a. $\frac{1}{2}x + \frac{3}{4}x$ Solution: $(\frac{1}{2} + \frac{3}{4})x$ or $\frac{5}{4}x$

b. $8x + 4y + 4 + 3x$ Solution: $8x + 3x + 4y + 4$ or $11x + 4y + 4$

c. $\frac{3}{4}x + 5y + \frac{1}{4}x$

d. $96a + 42b + 4a + 3b$

e. $\frac{2}{9} + \frac{1}{3}r + 3s + \frac{2}{3}r + 4s$

f. $2.4x + 6.8y + 4.9 + 1.6x$

g. $4(2x + 3x)$

h. $30a - 10a$

i. $2s - 4s$

j. $x + 3x - 4x$

k. $4(a + 1) - 4$

VI. Using the Distributive Property in Solving Simple Equations

A. Suggested Procedure

1. Have pupils review:

   Meaning of equations
   Equivalent equations
   Solution or root of an equation; solution set
   Principles of equation solving - Addition, Subtraction, Multiplication, Division

   Note: See Chapter X of Mathematics Grade 8, Curriculum Bulletin 1961-1962, Series No. 4

2. Have pupils consider the equation $3a + 5a = 16$.

   Have them recall that the distributive property may be used to obtain a simplification of the left member. That is, we may write

   $$\begin{aligned} (3 + 5)a &= 16 \quad \text{or} \\ 8a &= 16 \\ a &= 2 \quad \text{(Division principle or multiplication by reciprocal)} \end{aligned}$$

   2 is the root of the equivalent equation, $a = 2$, whose solution is obvious.

   The solution set is $\{2\}$.

   Have pupils check to see whether 2 is a root of the given equation $3a + 5a = 16$.

   Check

   $$\begin{aligned} 3 \cdot 2 + 5 \cdot 2 &\overset{?}{=} 16 \\ 6 + 10 &\overset{?}{=} 16 \\ 16 &= 16 \quad \text{True} \end{aligned}$$

3. Have pupils use the principles of equation-solving to think through the solution of an equation which uses more than one principle.

   a. Consider the equation $3x + 4 = 22$.

      Have pupils observe that more than one operation is needed to find the simplest equivalent equation in which the root is obvious. Have them see that both subtraction (addition) and division (multiplication) are needed.

Thus, $3x + 4 = 22$

$3x = 18$ (Subtraction principle or addition of additive inverse)

$x = 6$ (Division principle or multiplication by reciprocal)

Therefore, 6 is the root of the equivalent equation, $x = 6$, whose solution is obvious. The solution set is $\{6\}$.

Have pupils check to see whether 6 is a root of the given equation $3x + 4 = 22$.

b. Have pupils consider whether it makes any difference if the subtraction (addition) principle is applied before or after the division principle. To determine this, have them solve the equation $3x + 4 = 22$, by reversing the order in which the principles are applied.

Thus,

$3x + 4 = 22$

$\frac{3x + 4}{3} = \frac{22}{3}$ (Division Principle)

$\frac{3x}{3} + \frac{4}{3} = \frac{22}{3}$ (Distributive property) or $x + \frac{4}{3} = \frac{22}{3}$

$x = \frac{18}{3}$, or 6 (Subtraction Principle)

Have pupils observe that the same root was obtained as before. Have them discuss which order of operations is easier for solving equations of this kind.

c. Have pupils use a similar procedure to find the root of each of the following:

1) $2x + 4 = 10$

2) $3y - 9 = 15$

3) $7a + 10 = 59$

4) $\frac{1}{2}b + 8 = 14$

5) $\frac{1}{3}n - 4 = 9$

6) $1.5x + 3 = 7.5$

4. Have pupils consider the equation $2x + 4x + 8 = 38$.

Help them organize the solution of the equation as follows:

$2x + 4x + 8 = 38$

$(2+4)x + 8 = 38$ (Distributive property), or $6x + 8 = 38$

$6x = 30$ (Subtraction principle or addition of additive inverse)

$x = 5$ (Division principle or multiplication by reciprocal)

Therefore, 5 is the root of $x = 5$. The solution set is $\{5\}$.

Check (to see whether 5 is a root of $2x + 4x + 8 = 38$)

$2 \cdot 5 + 4 \cdot 5 + 8 \overset{?}{=} 38$

$10 + 20 + 8 \overset{?}{=} 38$

$38 = 38$ True

5. Have pupils consider the equation $2(x + 1) + x = 8$.

The distributive, commutative, and associative properties may be used to obtain a simplification of the left member as follows:

$2x + 2 + x = 8$ (Distributive property)

$2x + x + 2 = 8$ (Commutative property)

$3x + 2 = 8$ (Associative, Distributive)

$3x = 6$

$x = 2$

Therefore, 2 is the root of $x = 2$. The solution set is $\{2\}$.

Have pupils check to see whether 2 is a root of the given equation $2(x + 1) + x = 8$.

B. Suggested Practice

Solve the following equations by writing simpler equivalent equations. Check your solutions.

1. $2x + 3x = 20$
2. $8y - 5y = 33$
3. $11a + a = 48$
4. $\frac{1}{3}x + \frac{1}{6}x = 4$
5. $\frac{r}{2} - \frac{r}{4} = -3$
6. $.5a + .2a = 1.4$
7. $16y + 2y + 3 = 39$
8. $44 + 9x - 3x = 8$
9. $.1x + .9x + 8x = 81$
10. $\frac{3}{5}x + 3 - \frac{2}{5}x = 10$
11. $9y + 2y - 10 = 12$
12. $4(a + 1) + a = 19$
13. $3(x - 2) + x = 14$
14. $2y + \frac{1}{2}(2y - 4) = 7$
15. $.04x + .06(1500 - x) = 80$

VII. Solution of Verbal Problems

A. Suggested Procedure

Note: Help pupils develop skill in solving problems. Before doing any problem where letters are involved, have the pupils solve arithmetic problems involving the same situation.

Problems are introduced at this time because the techniques needed for their solution have now been developed. However, it is intended that the teacher should distribute the work in problem solving throughout the remainder of the course.

1. Review translation of English phrases into algebraic form. (See Chapter III, Section V)
2. Help pupils think through and express relationships involving number problems as follows:
   a. One number is 14. Another is twice the first. What is the second number? What is the sum of the numbers?

      One number is x. Another is twice the first. The sum of the numbers is 45. What is the second number in terms of the first? Express the sum of the numbers in terms of x.

b. Two times a number is subtracted from eight times the number, increased by 3. The difference is 27. What is the number?

Illustrative Solution

Let x = the number
$2x$ = two times the number
$8x + 3$ = eight times the number, increased by 3
$(8x + 3) - 2x = 27$ (the difference is 27)

$$\begin{aligned} 8x - 2x + 3 &= 27 \\ 6x + 3 &= 27 \\ 6x &= 24 \\ x &= 4 \end{aligned}$$

The number is 4.

Check result with the problem. If the number is 4, then two times the number is 2×4 or 8. Eight times the number increased by 3 is $8 \times 4 + 3$ or 35. The difference is $35 - 8$ or 27. True.

c. Have pupils practice similar problems.

1) A number increased by 3 times itself equals 8.4. What is the number?

2) Ann weighs 7 pounds more than Mary. Their combined weight is 213 pounds. How much does each girl weigh?

3) Jane's mother is 9 times as old as Jane. Together their ages total 30 years. How old is each?

4) A family's vacation budget allows a certain amount for camping expenses, three times this amount for food, and four times this amount for travel. If the total spent is $320, how much is allowed for each item?

3. Help pupils think through and express relationships involving consecutive number problems as follows:

Note: Review Chapter II of Mathematics Grade 8, Curriculum Bulletin, 1961-62 Series, #4.

a. Develop meaning of consecutive numbers; consecutive even numbers; consecutive odd numbers.

b. Solve problems

1) Thirteen is the first of three consecutive numbers. What are the next two? What is the sum of the first and third?

The first of three consecutive numbers is x. Express the next two in terms of x. Express the sum of the first and third.

2) The sum of the first and last numbers of 3 consecutive numbers is 28. What are the numbers?

Illustrative Solution:

Let $x = $ the first number
$x + 1 = $ the second number
$x + 2 = $ the third number
$x + (x + 2) = 28$ (the sum of the first and third numbers)
$2x + 2 = 28$ (Distributive property)
$2x = 26$
$x = 13$

The first number is 13, the second 14, the third 15.

Check result with the problem.
Three consecutive numbers: 13, 14, 15.
The sum of the first and third $13 + 15 = 28$. True.

c. Have pupils practice similar problems.

1) The sum of two consecutive numbers is 89. What are the numbers?

2) The first of two consecutive numbers is 13. What is the next odd consecutive number? What is their difference?

3) Select any two consecutive odd numbers. What is their difference?

4) x represents an odd number. Express the next consecutive odd number in terms of x. Show that their sum must be even.

5) The sum of two consecutive even numbers is 50. What are the numbers?

6) The sum of three consecutive even numbers is 48. What are the numbers?

4. Help pupils think through and express relationships in verbal problems involving coins as illustrated:

a. Andy has 4 dimes and 2 more nickels than dimes (and no other money). How many coins does he have? (10) What is their value in cents?

Value of 1 dime is 1 x 10 or 10 cents
2 dimes is 2 x 10 or 20 cents
3 dimes is 3 x 10 or 30 cents
4 dimes is 4 x 10 or 40 cents

Value of 1 nickel is 1 x 5 or 5 cents
2 nickels is 2 x 5 or 10 cents
.
.
.
6 nickels is 6 x 5 or 30 cents

The value of the ten coins is 70 cents.

If Andy has x dimes and 2 more nickels than dimes, how would you represent the number of nickels? the total number of coins?

How would you represent the value in cents of the dimes?

Represent the value in cents of the nickels.

Represent the total value in cents of all his coins.

b. Marilyn has a collection of dimes and quarters. She has 8 quarters and twice as many dimes. How many coins does she have? What is the value (in cents) of all her coins?

If Marilyn has q quarters and twice as many dimes, how would you represent the number of dimes? the number of coins?

How would you represent the value in cents of the quarters; of the dimes?

How would you represent the total value (in cents) of all the coins?

c. A jar of coins contains twice as many nickels as dimes and three times as many quarters as dimes. The total value of the coins in the jar is $4.75. How many nickels are there in the jar?

Note: The teacher should elicit what the domain of the variable will be in the equation.

Illustrative solution:

Let x represent the number of dimes, 2x the number of nickels and 3x the number of quarters.

Then, 10x = value of the dimes (in cents)
5(2x) = value of the nickels (in cents)
25(3x) = value of the quarters (in cents)

$10x + 5(2x) + 25(3x) = 475$ (the total value is 475 cents)

$10x + 10x + 75x = 475$

$95x = 475$

$x = 5$

The number of dimes in the jar is 5.

The number of nickels is $2x$ or 2×5, or 10, and

The number of quarters is $3x$ or 3×5, or 15.

Check results with conditions in the problem.

If there are 5 dimes, there are 10 (twice as many) nickels, and 15 (three times as many) quarters.

The value of 5 dimes is 50 cents, the value of 10 nickels is 50 cents, and the value of 15 quarters is 375 cents.

Therefore, the total value is 475 cents or $4.75.

d. Have pupils practice similar problems.

1) Bob has $8 in nickels and dimes. He has twice as many dimes as he has nickels. How many coins of each kind does Bob have?

2) A jar full of pennies and dimes contains 3 times as many dimes as pennies. The total amount of money in the jar if $6.20. How many coins of each kind are there in the jar?

3) Larry has 2 more nickels than pennies. If he were to spend one of his nickels and three of his pennies, he would have 68 cents left. How many nickels and how many pennies does he have?

4) A piggy bank contained 3 fewer dimes than quarters and 8 more nickels than dimes. The total value of the coins was $5.95. How many dimes were there?

5) Amy has $4.08 in pennies, dimes and quarters in her coin bank. If the number of quarters is 4 more than the number of pennies and the number of pennies is 2 less than the number of dimes, find how many coins of each kind Amy has.

5. Help pupils think through relationships in verbal problems involving investments as follows:

Note: Review meaning of interest, investments, income, and i = prt.

a. Mr. Smith invests \$4000 at 3% and \$2500 at 5%. What is the total amount he invests? What is the income on the \$4000 investment? What is the income on the \$2500 investment? What is the total income on the investments? (income per year)

If Mr. Smith invests \$6500, part at 3% and the remainder at 5%, how would you represent the part he invests at 5%? How would you represent the part he invests at 3%? What is the income on the 3% investment? What is the income on the 5% investment? What is the total income?

b. A man invests \$10,000, one part at 6% and the other part at 2%. His total income on the investment for one year is \$320. How much has he invested at each rate?

Illustrative Solution

Note: Reasons are indicated for the steps in the solution of the equation. Teachers should not require their pupils to give a reason for each step in each problem.

If we let x designate the number of dollars invested at 6%, then

$10{,}000 - x$ designates the number of dollars invested at 2%
$.06x$ is the income on the 6% investment
$.02(10{,}000 - x)$ is the income on the 2% investment

$$\begin{aligned}
.06x + .02(10{,}000 - x) &= 320 \text{ (the total income is \$320)}\\
.06x + 200 - .02x &= 320 \text{ (Distributive property)}\\
.06x - .02x + 200 &= 320 \text{ (Commutative property)}\\
.04x + 200 &= 320 \text{ (Associative and distributive properties)}\\
.04x &= 120 \text{ (Subtraction principle)}\\
x &= 3000 \text{(Division principle)}\\
10{,}000 - x &= 7{,}000
\end{aligned}$$

The amount invested at 6% is \$3000.
The amount invested at 2% is \$7000.

Check results with conditions in the problem

The two amounts of \$3000 and \$7000 total the investment of \$10,000.

If \$3000 is invested at 6%, then the income on this part of the investment is .06 of \$3000, or \$180.

If \$7000 is invested at 2%, then the income on this part of the investment is .02 of \$7000, or \$140.

Then the total income is \$180 + \$140, or \$320. All conditions of the problem have been checked.

c. Have pupils practice similar problems.

1) A man has loaned $4400 to two persons. His income from the two investments is $192. One note (a written promise to pay) bears interest at 5%, and the other bears interest at 4%. What is the face of each note?

2) Tom's father has $18,500 invested in two pieces of rental property. One yields 12% on the investment, and the other yields 10%. His total income from the two properties is $2080. How much is invested in each property?

3) One sum of money is invested at $4\frac{1}{2}$%; a second sum that is twice as large as the first sum is invested at 3%. The total interest from the sums is $210. How much is invested at each rate?

6. Help pupils think through relationships in verbal problems involving geometric figures as follows:

Note: It is helpful to use diagrams for geometric problems. The domain of a variable representing a geometric length is the set of positive numbers.

a. The length of a rectangle is four inches more than its width. If the width is 3 inches, what is the length? What is the perimeter?

If the width of a rectangle is represented by x and the length is four more than its width, how would you represent its length? How would you represent its perimeter?

b. The width of a rectangle is 5 inches. The length of the rectangle is 3 less than twice its width. What is the length of the rectangle? What is its perimeter?

If the width of a rectangle is represented by w, and its length is 3 less than twice its width, how would you represent its length? How would you represent its perimeter?

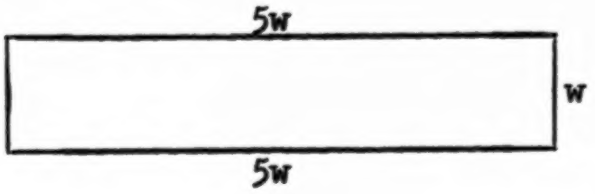
c. One leg of an isosceles triangle is 7 inches in length. The base is 3 inches longer than the leg. What is the length of the base? What is the length of the other leg? What is the perimeter of the triangle?

If the length of one leg of an isosceles triangle is represented by x inches, and the base is 3 inches longer than the leg, how would you represent the length of the base? How would you represent the length of the other leg? the length of the perimeter?

d. The length of a rectangle is 5 times its width. The perimeter is 48 inches. Find the dimensions of the rectangle.

Illustrative Solution

Let w be the number of inches in the width of the rectangle. Then the number of inches in the length is $5w$. Let us draw a picture to help us analyze and solve the problem:



The number of inches in the perimeter is $w + 5w + w + 5w$.

Then, $w + 5w + w + 5w = 48$

$$12w = 48 \text{ (Distributive property)}$$

$$w = 4$$

Therefore, the rectangle is 4 inches wide and 20 inches long.

Check results with conditions in the problem

The length is 5 times the width because $20 = 5 \times 4$.

The perimeter is $20 + 4 + 20 + 4$ or 48 inches.

e. Have pupils practice similar problems.

1) How long and how wide can a garden be made if it is to be 20 feet longer than it is wide, and its perimeter is to be 680 feet?

2) A triangle that draftsmen find useful has one angle that is twice as large as the first angle, and the third angle 30° more than the second. How large are the angles?

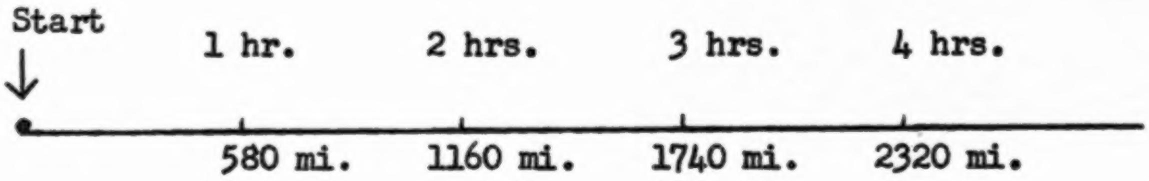
3) A rectangle has a pleasing appearance if its width is a little more than one-half its length. What dimensions will satisfy this condition if the width is 5 feet more than one-half the length, and the perimeter is 310 feet?

7. Help pupils review relationships in problems involving distance, rate, and time.

   a. Have pupils determine distance a boy walks in 4 hours if he walks 3 miles per hour. From this and similar problems, the pupil should gain an understanding of d = rt (distance formula).

   b. Have them practice solving distance problems involving only 1 object or vehicle. (if a plane averages 580 miles per hour, how far will it go at this speed in 4 hours?)

      1) Have them solve problem as illustrated below:


Start
1 hr.
2 hrs.
3 hrs.
4 hrs.
580 mi.
1160 mi.
1740 mi.
2320 mi.

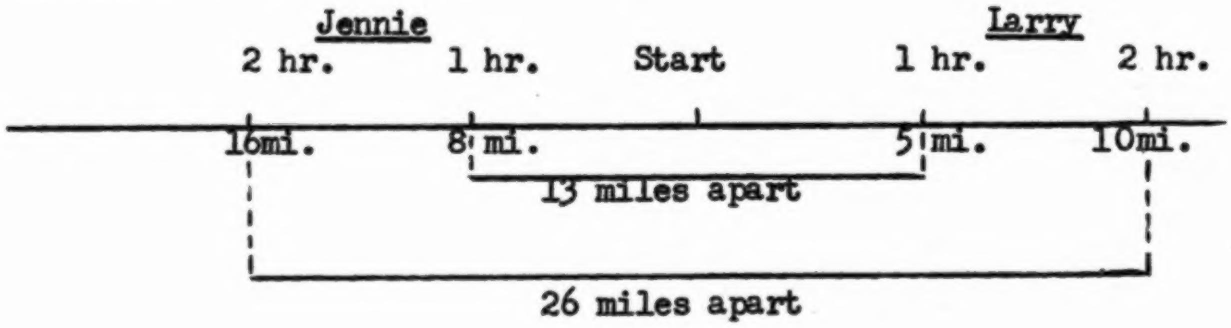
      2) Have them solve problem by formula:

         $d = rt$
         $d = 580 \times 4$
         $d = 2320$

   c. Have them solve distance problem involving 2 objects or people.

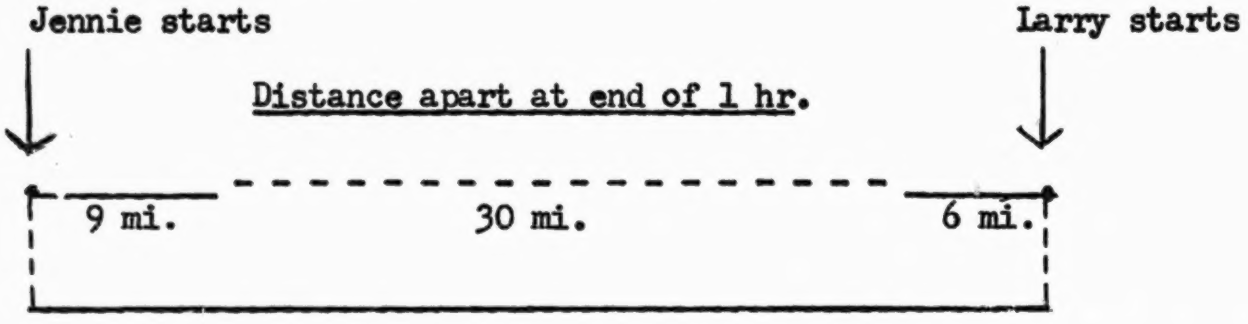
      1) Larry and Jennie on bicycles start from the same place at the same time and ride in opposite directions. Larry races at the rate of 5 mph and Jennie at 8 mph. How far apart are they in one hour? in 2 hours? in 4 hours?

         Encourage the pupils to use a diagram:


Jennie
Larry
2 hr.
1 hr.
Start
1 hr.
2 hr.
16mi.
8 mi.
5 mi.
10mi.
13 miles apart
26 miles apart

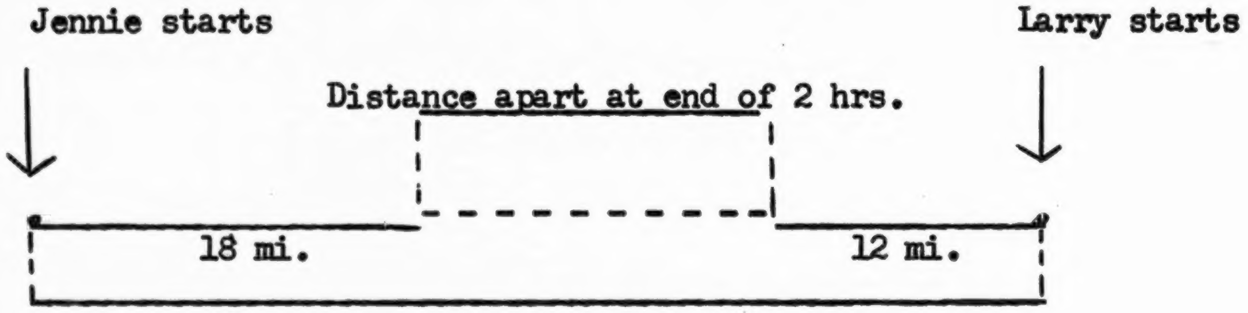
2) Two cars start from the same place at the same time and travel in opposite directions. One travels at 40 miles per hour and the other at 30 miles per hour. What distance will the first car have traveled in x hours? What distance will the second car have traveled in the same time? They are 350 miles apart in x hours. Express this fact in terms of x.

3) Jennie and Larry are 45 miles apart. They start toward each other at the same time. Jennie rides at an average speed of 9 mph and Larry rides at an average speed of 6 mph. In how many hours after they start will they meet?


Jennie starts
Larry starts
Distance apart at end of 1 hr.
9 mi.
30 mi.
6 mi.

Distance apart at start: 45 mi.

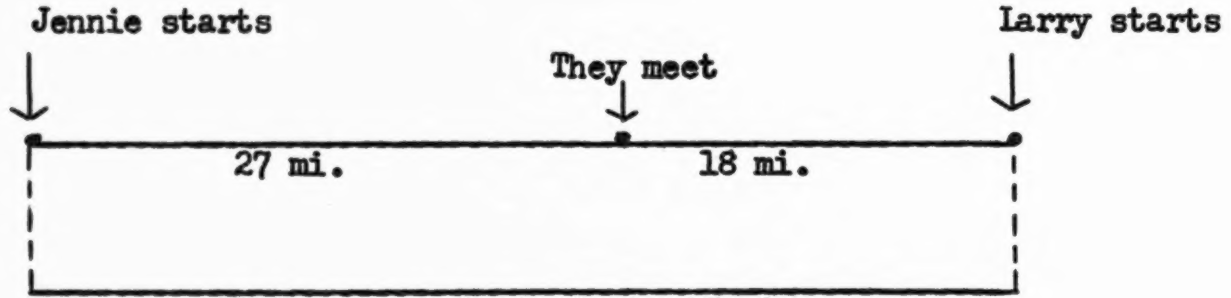
How much distance did they cover **together** in one hour? 9 + 6 = 15 mi.


Jennie starts
Larry starts
Distance apart at end of 2 hrs.
18 mi.
12 mi.

Distance apart at start: 45 mi.

How much distance did they cover together in 2 hrs.? 18 + 14 = 30 mi.

Complete the problem. They meet in 3 hours.


Jennie starts
Larry starts
They meet
27 mi.
18 mi.

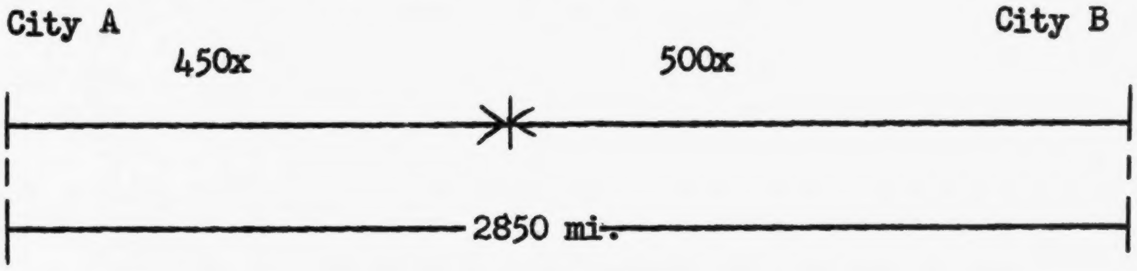
Distance apart at start: 45 mi.

4) Plane A starts from City A for City B traveling at 450 miles per hour. Plane B starts at the same time from City B for City A traveling at 500 miles per hour. In how many hours will they pass each other if the distance between the cities is 2850 miles?

Let x = number of hours it takes for the planes to meet. Each plane travels the same number of hours.

Distance plane A travels in x hours is 450•x

Distance plane B travels in x hours is 500•x



The facts in the problem may be organized in table form:

Rule	Rate (m.p.h.) x	Time (hr.) =	Distance (mi.)
Plane A	450	x	450x
Plane B	500	x	500x

Set up equation:

$$\begin{aligned} 450x + 500x &= 2850 \\ 950x &= 2850 \\ x &= 3 \end{aligned}$$

They will pass each other in 3 hours.

Check results with conditions in the problem.

The plane from City A traveling at 450 miles per hour will cover 1350 miles in 3 hours. The plane from City B traveling at 500 miles per hour will cover 1500 miles in 3 hours. Together they will cover 1350 + 1500 or 2850 miles in 3 hours.

d. Have pupils practice solving problems.

1) Two trains are traveling on parallel tracks toward each other. The slower one travels at 40 miles per hour, the faster at 60 miles per hour. What distance do the two trains cover in one hour? in two hours? If they start at the same time from towns which are 500 miles apart, in how many hours will they pass each other?

2) Frank and Sam leave school for home. They go in opposite directions. Frank walks at a rate one mile an hour faster than Sam, who walks at 3 miles per hour. In how many hours will they be 3.5 miles apart, the distance between their homes?

3) A bus leaves Boston for New York City 250 miles away traveling at 35 miles per hour. At the same time, another bus traveling along the same route leaves New York City for Boston traveling at 40 miles per hour. In how many hours will they pass each other?

4) Other problems may be found in any 9th year textbook.

8. Help pupils think through relationships in verbal problems involving mixtures as follows:

a. If a certain grade of coffee costs 75 cents a pound, how much will 10 pounds cost? 15 pounds?

b. What is the total cost of 20 pounds of 75-cent coffee and 35 pounds of 80-cent coffee?

c. How would you express the cost of x pounds of 75-cent coffee? of (50-x) pounds of 90-cent coffee?

d. Of 100 pounds of tea, x pounds are sold at 55 cents a pound and the remainder at 68 cents a pound. Express the number of pounds sold at 68 cents a pound. Express the amount of money received for the cheaper tea; the more expensive tea.

e. A grocer has two kinds of coffee, one priced at 80 cents a pound and the other at 70 cents a pound. How many pounds of each should be used to make a mixture of 120 pounds to sell at 74 cents a pound?

Illustrative solution

Guide pupils to see that the basic assumption of the situation is that the grocer receives the same amount of money after mixing the coffee that he would have received without mixing it.

Let x = number of pounds of 80-cent coffee

$120-x$ = number of pounds of 70-cent coffee

$80x$ = value of x pounds of 80-cent coffee (in cents)

$70(120-x)$ = value of $120-x$ pounds of 70-cent coffee (in cents)

Have pupils see that $80x + 70(120-x)$ represents the value (in cents) of the coffee without making a mixture. Have them see that the amount of money (in cents) received after mixing the two kinds of coffee is $74(120)$.

Then $80x + 70(120 - x) = 74(120)$

$80x + 8400 - 70x = 8880$ (Distributive property)

$80x - 70x + 8400 = 8880$ (Commutative property)

$10x + 8400 = 8880$ (Distributive property)

$10x = 480$ (Subtraction principle)

$x = 48$ (Division principle)

$120 - x = 72$

The grocer needs to mix 48 pounds of 80-cent coffee with 72 pounds of 70-cent coffee.

Check results with conditions in the problem

The two amounts, 48 pounds and 72 pounds, total 120 pounds.

The money received for selling 48 pounds of 80-cent coffee and 72 pounds of 70-cent coffee is $.80(48) + .70(72)$ or \$38.40 + \$50.40 or \$88.80. The value of 120 pounds of 74-cent coffee is $.74(120)$ or \$88.80. All conditions of the problem have been checked.

f. Have pupils practice similar problems.

1) The G.O. store at a certain school sold 240 notebooks the first day of school, some at 30 cents each and the rest at 45 cents each. If a total of \$85.50 was collected, how many of each kind were sold?

2) Sally bought some five-cent stamps and some two-cent stamps for \$1.79. She bought 55 stamps altogether. How many of each kind did she buy?

3) A confectioner mixes two kinds of candy obtaining a mixture of 30 pounds worth 40 cents per pound. If the costs per pound of the two kinds of candy are 50 cents and 35 cents, find how many pounds of each kind he should take.

B. Suggested Practice

Problems similar to those in section VII-A-2c, 3c, 4d, 5c, 6e, 7d, and 8f.

CHAPTER V

POLYNOMIALS

In this section the teacher will find suggestions for procedures to help pupils develop an understanding of the meaning of a polynomial, and understanding and skill in operations with polynomials: addition, multiplication, subtraction.

I. Meaning of Polynomials

A. Suggested Procedure

1. Have pupils review the meaning of base, exponent and power. (See Chapter III)

   Consider 5^2. What is the base, exponent, and power?

2. Have pupils consider expressions such as $2x + 2$ and $4x^2 + 1$; guide them to see that these expressions cannot be the names of numbers since they involve variables. Have pupils see the need for investigating the set of all such expressions.

3. Have pupils recall that each time a new set of numbers was constructed, it was done by expanding an old one, e.g., the set of fractions includes the whole numbers, the set of signed numbers includes the numbers of arithmetic (positive numbers), etc.

   We are going to construct a set of expressions such as $2x + 2$ and $4x^2 + 1$. The new set will contain all the numbers already worked with as well as expressions like $2x + 2$ and $4x^2 + 1$. Every member of this set will be called a polynomial.

4. Have pupils construct polynomials.

   a. Have them select any base x and write some powers of x (the exponents are limited here to the positive integral values).

      x, x^2, x^3...where a specific number may be substituted for x.

      Note: The letter in our polynomials will be considered, for our purpose, a variable with a domain.

b. They may now select any numbers they know

$\{\ldots -5, \ldots -3\frac{1}{2}, \ldots 0, \ldots 1, \ldots\}$ and use these to multiply with

$x, x^2, x^3 \ldots$.

They will create expressions such as the following:

$7 \cdot x$ or $7x$; $12 \cdot x^3$ or $12x^3$; $-2 \cdot x^4$ or $-2x^4$

c. Have pupils take some of their creations and add them. They may also add some of the numbers. Thus, they may have:

$7x + 12x^3$; $-2x^4 + 17$; $3x + 4x^2 + (-3x^3) + (-7)$

Tell pupils that any such sum is called a polynomial.

Have them suggest other polynomials.

5. Have pupils consider the polynomial $4x^2 + 3x + 6$. What is contained in it?

a. It has the single variable x, which has some set as its domain.

b. It contains the numbers 4, 3, and 6. These numbers are called the numerical coefficients of the polynomial (usually called simply the coefficients).

c. Certain operations are used: we multiply 4 and x^2 and 3 and x; we add $4x^2$, 3x and 6. Each of the three expressions that are added is called a term of the polynomial. The coefficient 6 is given a special name. It is called a constant term.

d. Have pupils examine the polynomial $-2x^3 + 6x^2 + 2x + 1$. They observe that this, too, contains a variable x which has some set as its domain. It contains some coefficients (-2, 6, 2, 1). The variable and coefficients and terms are combined by the operations of addition and multiplication.

e. Guide pupils to see that every polynomial (in one variable) has the same three characteristics: variable, coefficients, and operations.

6. Develop with pupil the understanding that a polynomial (in x) is an expression that can be constructed from the variable x and from some coefficients by means of addition and multiplication.

The variable in a polynomial need not be x. For example, $4y^2 + 6$ is a polynomial in y, and $3a^2 + 2a + 5$ is a polynomial in a.

7. Guide pupils to observe the following about notation:

   a. We usually do not write a coefficient of 1. For example, we write $4x^2 + x$ instead of $4x^2 + 1x$.

   b. When a term has a negative coefficient, we do not write a plus sign to indicate addition. Thus, instead of writing $3x^2 + (-2)x + (-6)$, we write $3x^2 - 2x - 6$.

8. Have pupils examine the following to see if they are polynomials:

$5x^2 + 3x - 7$	This is a polynomial.
$\frac{1}{2}x^2 + \frac{3}{4} + 8$	This is a polynomial.
9	This is a polynomial for the set of polynomials is an expansion of the set of directed numbers.
$\frac{3}{x} + 9$	This is not a polynomial since it involves division by x.

9. Develop with pupils an understanding of the degree of a polynomial.

   a. Have pupils consider the two polynomials:

   $2x^2 - 3x + 2$

   $2x^5 - 3x + 2$

   Have them compare the coefficients and exponents of comparable terms. How are they different? The first one is said to be of degree 5, the second one, of degree 2.

   b. Have pupils evaluate each polynomial by replacing x by 3:

   $2x^2 - 3x + 2 = 2 \cdot 3^2 - 3 \cdot 3 + 2 = 11$

   $2x^5 - 3x + 2 = 2 \cdot 3^5 - 3 \cdot 3 + 2 = 479$

   They observe that the value of the polynomial of degree 5 is very different from that of degree 2.

   c. Develop with pupils the understanding that the degree of a polynomial is the number which is the highest exponent of the variable appearing in the polynomial.

$8x^3 + 4x - 9$ is of degree 3

$2y^7 - 8$ is of degree 7

$5 - 6a$ is of degree 1

8 is of degree 0

Note: The degree of a non-zero constant polynomial is defined to be zero.

10. Discuss with pupils the desirability of arranging the terms in some definite order when operating with polynomials.

    Consider the polynomial $3x^2 + 5x^4 - 2x^3 + 6 + x$. The terms may be rearranged in any order using the associative and commutative principles. However, it is customary to rearrange the terms of a polynomial in one of two ways:

    $5x^4 - 2x^3 + 3x^2 + x + 6$

    Have pupils look at the exponents. Reading from left to right they are: 4, 3, 2, 1. ($x^1 = x$ by definition)

    Since the exponents become progressively smaller, this arrangement is called "descending order."

    $6 + x + 3x^2 - 2x^3 + 5x^4$

    Reading from left to right, the exponents are: 1, 2, 3, 4. ($x = x^1$ by definition) Since the exponents become progressively larger, this arrangement is called "ascending order."

11. Tell pupils polynomials may contain two or more variables such as:

    a. $3x^4 + 2x^3y - \frac{1}{3}x^2y^2 - \frac{5}{6}xy^3 + y^4$

    b. $x + y$

    c. $x^2 - y^2$

    In a the polynomial is in descending order of exponents of x and in ascending order of exponents of y.

12. Have pupils consider the individual terms of a polynomial as, for example, the terms of the polynomial, $2x^3 + 6x^2$. The individual terms, $2x^3$ and $6x^2$, are called monomials.

    The number 2 in $2x^3$ is the coefficient of the monomial and the 3 in $2x^3$ is called the exponent of the monomial, or the exponent of x in the monomial. What is the coefficient and the exponent of the monomial x^4? of the monomial $-8x$? Guide pupils to see that the degree of a monomial in one variable is the number which is its exponent.

13. Guide pupils to see that every polynomial is either a monomial or a sum of monomials. If a polynomial is a sum of two monomials, it is called a binomial; if it is a sum of three monomials, it is called a trinomial.

$a^3 + 9$	$4 - y$	are binomials
$6y^2 + 14 - 10y$		is a trinomial

B. Suggested Practice

1. Which of the following are polynomials?

$6x^2 - 5$ $\qquad$ $4 + \frac{2}{x}$ $\qquad$ $\frac{1}{3}x - 8$ $\qquad$ 11

2. Write three polynomials in x; three polynomials in y.

3. Identify the various coefficients, exponents, constant terms, and degree of each of the following:

a. $3x^2 + 2 + 4x$
b. $y^3 + 2y + 6y^2 + 5$
c. $32 + a^2 - 7a$
d. $1 + a^2 - 2a + 5a^3$
e. $x^2 + \frac{1}{3}x - 9$
f. $r^3 - 2r^2 - 7$
g. $6 - 3x$
h. $-2a$
i. 12

4. Arrange each polynomial in #3 in descending order and in ascending order.

5. Evaluate the polynomial $3y^2 + y - 4$ for each of the following replacements for y:

2 $\qquad$ 0 $\qquad$ -4 $\qquad$ 1

II. Addition of Polynomials

A. Suggested Procedure

1. Review the use of the distributive property to simplify expressions.

a. $2x + 3x = (2 + 3)x$, or $5x$

$5a + 2a = (5 + 2)a$, or $7a$

$5a^2b + 7ab^2 = (5 + 7)ab^2$, or $12ab^2$

Have pupils recognize that this results in a procedure for adding monomials.

b. Does the distributive property help us to simplify the following expressions:

$3x + 2y$

$3b^2 + 2$

$5a^2b + 2ab^2$

Are $3x$ and $2y$ monomials in the same variable and of the same degree?

Are $3b^2$ and 2 monomials in the same variable and of the same degree?

Are $5a^2b$ and $2ab^2$ monomials with the same literal factors?

c. In addition of two (or more) monomials with the same literal factors, the sum can be expressed as a monomial through the use of the distributive property.

d. Have pupils circle the monomials in the same variable and of the same degree:

1) $2a$ $5c$ $-3a$ $7d$ $5a^2$

2) $3x^2$ $-2x$ 1 $5x^2$

3) 4 $3m$ -5 $3m^2$

Tell pupils the terms of an expression which have the same variables and the same exponents for corresponding variables are called similar terms or like terms.

e. When possible, write a monomial for each of the following:

1) $4x^2 + 7x^2$

2) $13n^3 + 8n^3$

3) $a - 6a$

4) $-3m + 4n$

5) $3x^2y + 8x^2y$

6) $7x + 3x^2$

2. Using the Commutative, Associative and Distributive properties to add polynomials of the first degree.

Note: Since a polynomial has been defined in terms of a variable, with some set as its domain, the number properties which hold for variable expressions will hold for polynomials as well. However, in higher mathematics a polynomial may be regarded as a form, with x as an arbitrary symbol to which no meaning is attached. If this meaning of a polynomial were adopted, it would then be necessary to establish the Commutative, Associative and Distributive properties for polynomials.

a. $(x + 7) + 2x = (x + 2x) + 7$ Commutative, Associative properties

$= (1 + 2)x + 7$ Distributive property

or $3x + 7$

$(4x + 2) + (3x + 5) = (4x + 3x) + (2 + 5)$ Commutative, Associative properties

$= (4 + 3)x + (2 + 5)$ Distributive property

or $7x + 7$

Have pupils observe that the sum of two polynomials in x is also a polynomial.

b. Have pupils check to see whether $(4x + 2) + (3x + 5)$ and $7x + 7$ name the same number when x is replaced by a particular value, say 2.

$$(4\cdot 2 + 2) + (3\cdot 2 + 5) \overset{?}{=} 7\cdot 2 + 7$$

$$10 + 11 \overset{?}{=} 14 + 7$$

$$21 = 21$$ True

3. Using Number Properties to add polynomials of degree greater than one.

a. $(2x^2 + 4x + 3) + (3x^2 + 5x + 6)$

$=(2x^2 + 3x^2) + (4x + 5x) + (3 + 6)$ Commutative, Associative properties

$=(2 + 3)x^2 + (4 + 5)x + (3 + 6)$ Distributive property

or $5x^2 + 9x + 9$

Similarly,

$(4x^3 - 6x^2 + x + 1) + (-2x^3 - 5x^2 + 2x - 2)$

$=(4x^3 - 2x^3) + (-6x^2 - 5x^2) + (x + 2x) + (1 - 2)$ Commutative, Associative properties

$=(4 - 2)x^3 + (-6 - 5)x^2 + (1 + 2)x + (1 - 2)$ Distributive property

or $2x^3 - 11x^2 + 3x - 1$

Have pupils realize that in adding polynomials of degree greater than one, we combine similar terms and thereby obtain a simpler but equivalent expression. This is accomplished by adding the numerical coefficients of similar terms.

b. Have pupils realize that it may be helpful to arrange the work vertically. For example, the sum of

$3a^2 + 4a + 5$ and $2a^2 + 3a$

may be arranged as:

$$\begin{aligned} &(3a^2 + 4a + 5) + (2a^2 + 3a) \\ &= (3a^2 + 2a^2) + (4a + 3a) + 5 \\ &= 5a^2 + 7a + 5 \end{aligned}$$

or

$$\begin{array}{r} 3a^2 + 4a + 5 \\ \underline{2a^2 + 3a \phantom{{}+5}} \\ 5a^2 + 7a + 5 \end{array}$$

c. Have pupils learn to check the addition of polynomials by assigning a numerical value to the variable and evaluating the expressions. For example,

$$\begin{array}{lcl} & & \underline{\text{Check: Let } a = 2} \\ 3a^2 + 4a + 5 & \longrightarrow & 3 \cdot 4 + 4 \cdot 2 + 5 = 25 \\ \underline{2a^2 + 3a} & \longrightarrow & 2 \cdot 4 + 3 \cdot 2 \qquad = \underline{14} \\ 5a^2 + 7a + 5 & \longrightarrow & 5 \cdot 4 + 7 \cdot 2 + 5 \overset{?}{=} 39 \\ & & 39 = 39 \quad \text{True} \end{array}$$

Discuss the disadvantages of selecting 1 or 0 as numbers for checking.

B. Suggested Practice

Add the following polynomials. Give the degree and coefficients of each result.

1. $(x^2 + 2x + 3) + (4x^2 + 6x + 1)$
2. $(5y^2 - 7y + 3) + (y^2 + 7y - 2)$
3. $(-3b^2 + 2b + 8) + (b^2 + 5)$
4. $(-6t + 4) + (2t^2 + 8t - 7)$
5. $(5r^2 + 2r + 12) + (-6r - 3)$
6. Refer to pages in various textbooks for additional examples.

III. Multiplication of Polynomials

A. Suggested Procedure

1. The product of monomials

a. Review with pupils use of the commutative and associative principles in simplification.

1) $3x \cdot 4x = 3 \cdot 4 \cdot x \cdot x$ Commutative and Associative properties

or $12x^2$ (superscript Notation)

2) $3y \cdot 2y \cdot 5xy = 3 \cdot 2 \cdot 5 \cdot x \cdot y \cdot y \cdot y$ Why?

or $30xy^3$ Why?

3) Have pupil practice multiplying monomials.

$2a \cdot 2b \cdot 6 \cdot ab$	$6r \cdot \frac{1}{2}r \cdot 8rs$
$5m \cdot 3n \cdot mn$	$4x \cdot (-3y) \cdot 5z$

b. Have pupils consider the meaning of products such as:

$a^2 \cdot a = (a \cdot a) \cdot a = a \cdot a \cdot a = a^3$

$x^5 \cdot x^2 = (x.x.x.x.x) \quad (x.x) = x.x.x.x.x.x.x = x^7$

Guide pupils to see that in multiplying two powers of the same base, the product has the same base. The exponent of the product is found by adding the original exponents.

Have pupils realize that this rule of exponents applies only when the bases of the powers are the same. It cannot be used, for example, to write a simplification of the product x^3y^2 because the bases of the powers x^3 and y^2 are different. This is known as the rule of exponents for multiplication.

c. Have pupils use the rule of exponents for multiplication together with the Commutative and Associative properties to express the product of monomials.

1) $(4x^2) \cdot (3x^4) = 4 \cdot 3 \ x^2 \cdot x^4$ Why?

$= 12x^6$ Why?

Have pupil examine $4x^2$. Is it a monomial?

Have pupil examine $3x^4$. Is this a monomial?

Have pupil examine the product $12x^6$. Is this a monomial?

The answer to all 3 questions is, "Yes."

Have pupils conclude that multiplying a monomial by a monomial results in a monomial.

2) $(x^2)(8x)(\frac{1}{3}x^2)(-3x^4) = 8\cdot\frac{1}{3}\cdot(-3)\cdot x^2\cdot x\cdot x^2\cdot x^4$
$= 8x^9$

Have pupils notice that the product of any number of monomials is also a monomial.

3) Have pupils consider the difference between the two expressions:

$3x^2$ and $(3x)^2$

$3x^2 = 3\cdot x\cdot x$

$(3x)^2 = 3x\cdot 3x = 3^2\cdot x^2 = 9x^2$

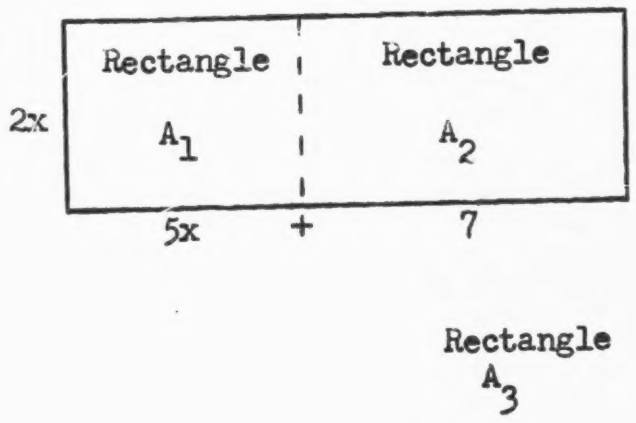
Have pupils note that in raising a product to a power each of the factors of the product is raised to that power.

d. Have pupils practice multiplying monomials.

1) $(3x^2)(7x^3)$
2) $(-20x^4)(6x)$
3) $(-10y^5)(-6y^3)$
4) $(-\frac{5}{3}a^2)(-\frac{7}{2}a^4)$
5) $(-3x)(-5x^2)(8x^5)$
6) $(4x)^2$
7) $(4xy)^2$
8) $(-3b)^3$

2. The product of polynomials by monomials

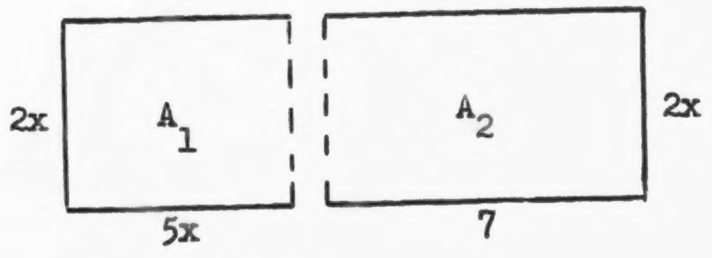
a. Have pupils examine the diagram at the right. They should notice that the area of the large rectangle (A_3) equals the sum of the areas of rectangles A_1 and A_2, or $A_3 = A_1 + A_2$.



b. Have them consider each of the smaller rectangles separately and find the area of each.

$A_1 = (5x)(2x) = 10x^2$

$A_2 = 7\cdot 2x = 14x$



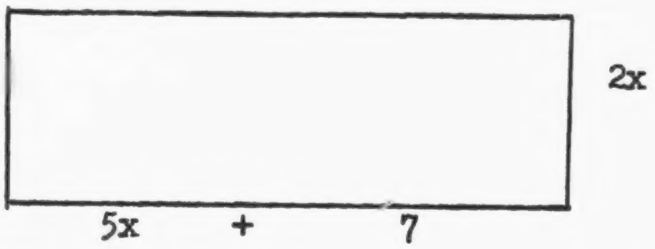
c. The pupils now find the area of the large rectangle by adding the areas of the other two rectangles.

$A_3 = A_1 + A_2 = 10x^2 + 14x$

d. Have pupils express the area of the large rectangle (A_3) in terms of its dimensions.

$A_3 = lw$

$A_3 = (5x+7)(2x) = (2x)(5x+7)$



Rectangle A_3

e. Have pupils discuss $(2x)(5x+7)$ and decide which property may be applied to polynomials to find an equivalent expression. It is the distributive property of multiplication over addition.

$(2x)(5x+7)=(2x)(5x) + (2x)(7) = 10x^2 + 14x$ Distributive property

They should check this with the area found in 2-c.

f. Have pupils substitute numerical values for x in $(2x)(5x+7)=10x^2+14x$. For the domain of x use $\{2, 3\}$. (Also substitute values of x in A_1, A_2, and A_3)

Have pupils determine whether or not true statements result.

g. After several similar examples, have pupils realize that to multiply a binomial by a monomial, we use the distributive property: we multiply each term of the binomial by the monomial and add these products.

h. Use a procedure similar to 2a-2g to have pupils learn to find the product of a monomial and any polynomial.

i. Have pupils practice finding products such as:

1) $2(5x + 4)$
2) $-3(2a + 1)$
3) $5x(x^2 + y)$
4) $-3x(2 - x)$
5) $x^2(7x - .4)$
6) $5x^3(2 - 4xy)$
7) $-7y^2(2y^2 + 3y)$
8) $-2(a^2 - 2a + a)$
9) $b(6b^2 + 4b - 8)$
10) $-3x^4(x^4 - 11x + 6)$

j. Have pupils practice solving equations such as:

1) $2x + 3(x + 2) = 16$
2) $4n + 3(n - 1) = 18$
3) $-2a + 5(a + 1) = 20$
4) $2y + .5(3 - y) = 4.5$
5) $(2x + 4) + 3(x - 1) = 11$
6) $10(x - 2) + 4(2x - 1) = 12$

k. Have pupils practice solving problems such as the following by representing each answer as a polynomial or as a directed number.

1) If x represents the length of a certain rectangle and if $2x + 7$ represents its width, how would you represent its area?

2) A man drove for 3 hours at $(x + 30)$ miles per hour, then at $(x + 40)$ miles per hour for 6 hours. Represent the total distance traveled.

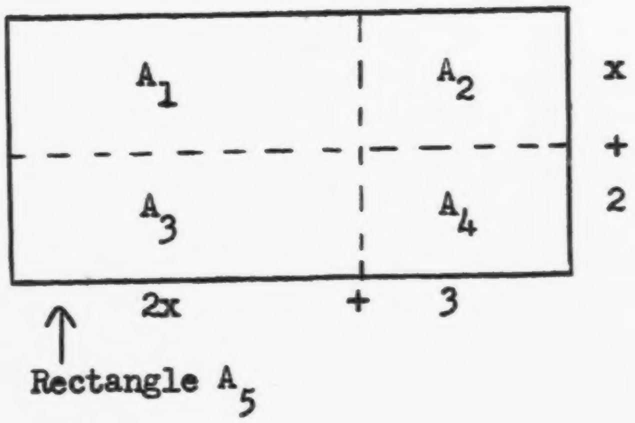
3) An automobile traveled at x miles per hour for 5 hours, then at $(x + 15)$ miles per hour for 2 hours. It traveled a total distance of 310 miles. What was its rate during the five-hour period? the two-hour period?

4) Mary earned y dollars on each of three days and $(y - 1)$ dollars on each of the following two days. How would you represent her total earnings?

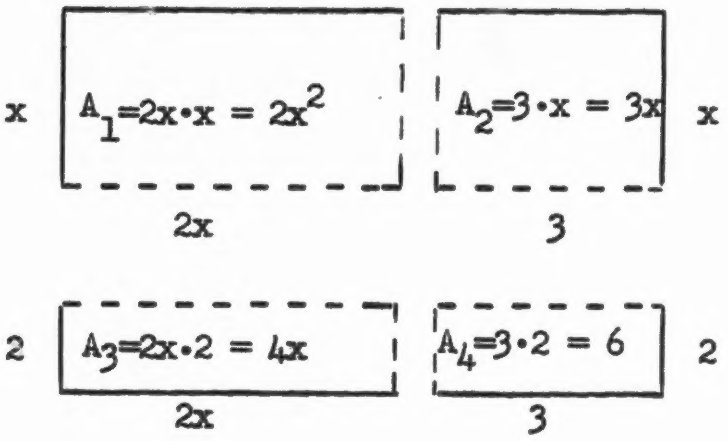
5) In the preceding problem, if Mary's earnings total $23, what were her daily earnings?

3. The product of two binomials

a. Have pupils examine the diagram at the right. They should notice that the area of the large rectangle (A_5) equals the sum of the areas of rectangles A_1, A_2, A_3 and A_4.



b. Have them consider the smaller rectangles separately and find the area of each.



c. The pupils now find the area of the large rectangle.

$$\begin{aligned} A_5 &= A_1 + A_2 + A_3 + A_4 \\ &= 2x^2 + 3x + 4x + 6 \\ &= 2x^2 + 7x + 6 \end{aligned}$$

d. Have pupils express the area of the large rectangle in terms of its dimensions.

$$\begin{aligned} a &= lw \\ &= (2x + 3)\ (x + 2) \end{aligned}$$

e. Have pupils discuss $(2x+3)(x+2)$ and decide which property may be applied to polynomials to find an equivalent expression. By first treating $2x + 3$ as a number by which the binomial $x + 2$ is multiplied, we obtain:

$(2x+3)(x+2) = (2x+3)(x) + (2x+3)2$	Distributive property
$= 2x^2 + 3x + 4x + 6$	Why?
$= 2x^2 + 7x + 6$	Why?

Have pupils check this with the area found in 3-c.

f. Have pupils illustrate geometrically the product $(x+2)(x+3)$ and find an equivalent expression for the product.

g. Have pupils multiply using the associative and distributive properties as reasons.

Sample: $(x+2)(x-1)$

Expression	Reasons
1) $(x+2)(x-1) = (x+2)x + (x+2)(-1)$	Why?
2) $= x^2 + 2x - x - 2$	Why?
3) $= x^2 + (2x - x) - 2$	Why?
4) $= x^2 + (2 - 1)x - 2$	Why?
5) $= x^2 + x - 2$	

Have pupils realize that the distributive property of multiplication over addition is used several times to find a single polynomial that is the product of two binomials.

h. Have pupils decide whether or not the binomials $(x+2)(x-1)$ would yield the same product, $x^2 + x - 2$, as $(x-1)(x+2)$. Check by carrying out the multiplication.

i. Have pupils learn to multiply the binomials when they are in a vertical arrangement.

1) From 3-e, $(2x+3)(x+2) = 2x^2 + 7x + 6$

2) Have pupils let $x = 10$ and examine the following:

If $x = 10$, $2x + 3 = 23$, $x + 2 = 12$, and $2x^2 + 7x + 6 = 276$

$$\begin{array}{lrcrcr}
 & 23 & = & 20 + 3 & = & 2(10) + 3 \\
 & \underline{\times\ 12} & = & \underline{\times 10 + 2} & = & \underline{\times\ \ 10\ \ + 2} \\
(2\times 23) & 46 & & 40 + 6 & & 4(10) + 6 \\
(10\times 23) & \underline{23} & & \underline{200 + 30\phantom{{}+6}} & & \underline{2(10)(10) + 3(10)\phantom{{}+6}} \\
 & 276 & & 200 + 70 + 6 & & 2 \cdot 10^2 + 7(10) + 6
\end{array}$$

3) Now have pupils follow the pattern in i-2) and multiply the binomials from right to left:

$$\begin{array}{rl}
2x + 3 & \\
\underline{x + 2} & \\
4x + 6 & 2(2x + 3) \\
\underline{2x^2 + 3x\phantom{{}+6}} & x(2x + 3) \\
2x^2 + 7x + 6 &
\end{array}$$

4) Tell pupils that in multiplication of binomials it is customary to proceed from left to right.

$$\begin{array}{rrrl}
 & 2x & +\ 3 & \\
 & x & +\ 2 & \\ \cline{1-3}
2x^2 & +\ 3x & & x(2x + 3) \\
 & +\ 4x & +\ 6 & 2(2x + 3) \\ \cline{1-3}
2x^2 & +\ 7x & +\ 6 &
\end{array}$$

j. Have pupils practice multiplication of binomials. Use both horizontal and vertical form. Have them check the accuracy of the multiplication by evaluating the factors and their product, using any numbers except 0 and 1.

Guide pupils to see that multiplying two binomials will result in four partial products. Since similar terms may be combined, the final product may contain fewer than four terms.

k. Have pupils realize that we use a similar procedure to multiply polynomials which contain three or more terms. They should realize that the number of terms in the multiplicand times the number of terms in the multiplier determines the number of partial products. For example, how many partial products are there when the multiplicand is $x^2 + 3x + 5$ and the multiplier is $x + 5$?

l. Have pupils try various examples so that they will discover:

1) The product of two monomials is a monomial.

2) The product of a monomial and a binomial is a binomial.

3) The product of two binomials containing similar terms is generally a trinomial.

   $(2x + 3)(x + 2) = 2x^2 + 7x + 6$

   Have them try to find a case in which the product of two binomials is a polynomial of four terms.

   Have them try to find a case in which the product of two binomials is a binomial.

   Have pupils realize the product of any two polynomials is a polynomial.

B. Suggested Practice

Express the product in each case as a monomial.

1. $(3x)(7x)$
2. $(2x)(5x^3)$
3. $(3x^2)(7x^3)$
4. $(-4y)(7y^2)(3)$
5. $(\frac{1}{3}x)(-4x^4)(x)$
6. $(5s^2)(\frac{1}{5})(-2s^3)$
7. $(2xy)(5x^2)$
8. $(.5a^2b)(2b^3)$
9. $(-7x^2)(2y)(-3y)$
10. $(2r)^3\ (3r)$

Have pupils perform the following multiplication. (Illustrate examples 11 and 12 geometrically.)

11. $(4a + 6)(3a)$
12. $(6b + 7)(3b)$
13. $(5y)(6y - 3)$
14. $(-3a)(2a + 3)$
15. $(\frac{x}{2})(4x - 1)$
16. $(.7)(2b + .3)$
17. $(\frac{1}{2}x)(4x - 6)$
18. Refer to textbook for additional practice.

Multiply the following polynomials using the horizontal form. Give a reason for each step.

19. $(x + 3)(x + 4)$
20. $(x - 2)(x + 6)$
21. $(2a - 1)(a - 2)$
22. $(3y + 2)(4y - 1)$
23. $(x + 5)(x - 2)$
24. Illustrate geometrically and then express the area of a rectangle whose dimensions are $(x + 5)$ and $(x - 2)$.
25. Check examples 19 and 20 by substituting 3 for x.

Multiply vertically:

26. $(a + 3)(a - 2)$

27. $(x + 5)(x - 3)$

28. $(2y + 3)(3y - 7)$

29. $(3m - 8)(2m - 6)$

30. $(3x - .7)(4x + .3)$

31. $(a + 3)(a - 3)$

32. $(x^2 - 3x + 1)(x + 7)$

33. $(.3x^2 + 2)(.4x^2 - 3)$

34. $(a + b)(a + b)$

35. $(c + d)(c - d)$

36. $(m + 3n)(m + n)$

OPTIONAL

Use the properties of numbers to prove that each of the following is a true statement for all values of the variable.

37. $(a + b)(c + d) = ac + bc + ad + bd$

38. $(x + y)(x + y) = x^2 + 2xy + y^2$

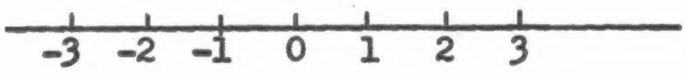
39. $(x + y)(x - y) = x^2 - y^2$

IV. Subtracting a Polynomial from a Polynomial

A. Suggested Procedure

1. Review the uses of the "-" sign.

a. Signed or directed number:



b. Sign of the binary operation of subtraction: $(8) - (5) = 3$.

c. Indicating additive inverse or opposite (or negative of)

Have pupils recall that the sum of a number and its additive inverse (opposite) is zero.

Thus, the additive inverse of 3 is -3 because their sum is zero. $-(-3) = ?$ $-(-3)$ means additive inverse of -3 which is 3.

2. Meaning of additive inverse of a polynomial

a. Have pupils review that subtraction of a number is performed by the addition of its additive inverse.

$(+6) - (+2) = (+6) + (-2)$ since -2 is opposite of $+2$

$(+3) - (-4) = (+3) + (+4)$ since $+4$ is opposite of -4

Thus, for any number a and b, $a - b = a + (-b)$.

The opposite of b is $(-b)$ where b may be positive, negative, or zero.

b. Have pupils consider what we shall mean by subtracting polynomials.

Note: Since the set of polynomials is an expansion of the set of signed numbers, we would like the meaning of subtraction of polynomials to parallel the meaning of subtraction of signed numbers. Therefore, let's agree that subtracting a polynomial shall mean adding its additive inverse.

c. Finding the additive inverse of any polynomial

1) Have pupils realize we need to know how to find the additive inverse of a polynomial. Does every polynomial in the set of polynomials have an additive inverse? For example, can we find a polynomial and add it to $4x$ and get a sum of zero?

Guide pupils to see that

$4x + (-4x)$

$= (4 - 4)x$ (Distributive property)

$= 0 \cdot x = 0$ (Multiplicative property of 0)

Then $4x$ and $-4x$ are additive inverses.

Have pupils practice finding the additive inverse of each of the following:

7 $6a$ $2x^2$ $-5y^2$

The additive inverse of a term of a polynomial is another term such that their sum is zero. Pupils should realize that to find the additive inverse of a term, the coefficient of the term is replaced by its additive inverse.

2) Have pupils observe that in the same way:

$(2x + 5) + (-2x - 5)$

$=(2x - 2x) + (5 - 5)$ Why?

$=(2 - 2)x + (5 - 5)$ Why?

$= 0$

Then $2x + 5$ and $-2x - 5$ are additive inverses.

$(-4y^2 + 3y) + (4y^2 - 3y)$

$= (-4y^2 + 4y^2) + (3y - 3y)$

$=(-4 + 4)y^2 + (3 - 3)y$

$= 0$

Then $-4y^2 + 3y$ and $4y^2 - 3y$ are additive inverses

$(3a^2 - 5a + 2) + (-3a^2 + 5a - 2)$

$=(3a^2 - 3a^2) + (-5a + 5a) + (2 - 2)$

$=(3 - 3)a^2 + (-5 + 5)a + (2 - 2)$

$= 0$

Then $3a^2 - 5a + 2$ and $-3a^2 + 5a - 2$ are additive inverses.

The additive inverse of a polynomial is another polynomial each of whose terms is the additive inverse of the corresponding term of the original polynomial.

Have pupils realize that to find the additive inverse of any polynomial, we replace each coefficient by the additive inverse of the coefficient.

Have pupils practice finding the additive inverse of each of the following:

$7x + 5$ $-6c + 4d$ $3b^2 + b - 2$ $-3x^2 - 12x - 1$

3) Have pupils recall that the additive inverse of a number may be indicated by writing a "-" in front of the number. The additive inverse of a polynomial may be indicated in the same way.

Polynomial	Its Additive Inverse
$3x - 2$	$(-3x + 2)$ or $-(3x - 2)$
$10x^2 - 5x + 3$	$(-10x^2 + 5x - 3)$ or $-(10x^2 - 5x + 3)$

Have pupils realize that whereas a polynomial has only one additive inverse, this inverse can be written in more than one way. Sometimes it is more convenient to have the inverse written in one way, sometimes in another.

3. Subtracting a polynomial

a. Guide pupils to subtract a polynomial from a polynomial, as follows:

1) $(5x + 7) - (4x - 2) = (5x + 7) + (-4x + 2)$ Why?

$= (5x - 4x) + (7 + 2)$ Why?

$= x + 9$ Why?

2) $(2x^2 + 7x - 8) - (6x^2 - 8x - 5) = (2x^2 + 7x - 8) + (-6x^2 + 8x + 5)$

$= (2x^2 - 6x^2) + (7x + 8x) + (-8 + 5)$

$= -4x^2 + 15x - 3$

3) $(4a^2 - 3a + 2) - (2a^2 + 7) = (4a^2 - 3a + 2) + (-2a^2 - 7)$

$= (4a^2 - 2a^2) - 3a + (2 - 7)$

$= 2a^2 - 3a - 5$

b. Have pupils perform the following subtractions.

1) $(3a - 2) - (4a + 3)$

2) $(6b + 3) - (5b - 7)$

3) $(4x^2 - 3y) - (7x^2 + 3y)$

4) $(2x^2y - 4x) - (3x^2y - 8x)$

5) $(3ab^2 - 4a^2b + 6ab) - (7ab^2 - 2a^2b + 6ab)$

6) $(4x^2y^2 - 2xy + 7) - (3x^2y^2 - 9)$

7) $(5a^3b^2 - 7a^2b^2 + 3ab) - (4a^2b + 6ab^2 - 7)$

c. Subtracting polynomials in vertical arrangement.

1) Have pupils subtract $2x^2 - 3x$ from $4x^2 + 5x$

2) Have them rearrange their example as follows:

Horizontal Form

$(4x^2 + 5x) - (2x^2 - 3x)$

Vertical Form

$$\begin{array}{rr} \text{Subtract:} & 4x^2 + 5x \\ & \underline{2x^2 - 3x} \end{array}$$

The vertical form

$$\begin{array}{ccc} \underline{\text{Subtract}} & & \underline{\text{Add}} \\ 4x^2 + 5x & \text{becomes} & 4x^2 + 5x \\ \underline{2x^2 - 3x} & & -\ \underline{2x^2 + 3x} \\ & & 2x^2 + 8x \end{array}$$

3) Have pupils check this example by substituting a numerical value for x in the minuend, subtrahend, and difference.

For example, when $x = 2$

$$\begin{array}{rcccll} 4(2)^2 + 5(2) & = & 4(4) + 10 & = & 26 & \text{Minuend} \\ \underline{2(2)^2 - 3(2)} & = & \underline{2(4) - \ \ 6} & = & \underline{\ 2} & \text{Subtrahend} \\ 2(2)^2 + 8(2) & = & 2(4) + 16 & \overset{?}{=} & 24 & \text{Difference} \\ & & 8 \ \ + 16 & \overset{?}{=} & 24 & \\ & & 24 & = & 24 & \text{True} \end{array}$$

4) Have pupils also check by adding the difference and the subtrahend to see whether the sum is the same as the minuend.

$$\begin{array}{lrl} \text{Add} & 2x^2 - 3x & \text{Subtrahend} \\ & \underline{2x^2 + 8x} & \text{Difference} \\ & 4x^2 + 5x & \text{Minuend} \end{array}$$

5) Have pupils practice subtracting the following polynomials vertically. Check each example in two ways.

a) $\begin{array}{r} 3a + b \\ \underline{2a + b} \end{array}$ b) $\begin{array}{r} -4c + 7d \\ \underline{-3c - 4d} \end{array}$ c. $\begin{array}{r} 6b^2 - 8c \\ \underline{4b^2 + 6c} \end{array}$ d) $\begin{array}{r} 3x^2 - 4x - 2 \\ \underline{2x^2 - 4x + 9} \end{array}$

e) From the sum of $(3x^2 + 2x)$ and $(2x^2 - 3x)$ subtract $(6x^2 - x)$

f) Other examples from textbook.

B. Suggested Practice

1. What is the additive inverse (opposite) of each of the following?

   a. $-3x$

   b. $2y^2$

   c. $3y - 4$

   d. $6x^2 - 7$

   e. $a^2 - b^2 - c^2$

   f. $6x^2 - 3x + 4$

2. $6x - (2x) = ?$

3. $7n^2 - (-3n^2) = ?$

4. $-4x - (-8x) = ?$

5. $(7x + 2x) - 4x = ?$

6. $(8a^2 - 7a) - (-3a) = ?$

7. $(.2x^2 - x) - (.3x^2) = ?$

8. $(\frac{1}{4}y^2 + \frac{1}{2}y) - (-\frac{1}{2}y) = ?$

9. $(6y^2 - 7z) - (+3y^2) = ?$

10. Do in horizontal form:

    a. What must be added to $3x^2 - 2x$ to give $5x^2 - 3x$?

    b. Subtract $3x^2 - 4x$ from $4x^2 - 3x$.

    c. From $2a^2 - 3a + 4$ subtract $2a^2 - 2a - 3$.

    d. Subtract $2x^2 - 3x + 4$ from $2x - 1$.

    e. Subtract $3x^2 - 2x + 2$ from $4x^2 - x + 2$.

    f. Subtract $5b^2 + 7b$ from $9b^2 - 8b$.

    g. Select other examples from textbook.

11. Do in vertical form:

    Check by evaluation: $x = 2$, $a = 3$, $b = 4$, $c = 5$

    Subtract

    a. $$\begin{array}{r} 4x^2 - 3x \\ \underline{2x^2 + 5x} \end{array}$$

    Subtract

    b. $$\begin{array}{r} 3x^2 - 7x - 4 \\ \underline{2x^2 - 3x + 3} \end{array}$$

    Subtract

    c. $$\begin{array}{r} 8a - 6b - 2c \\ \underline{-3a + 6b - 2c} \end{array}$$

12. Other examples from textbook.

13. Solve each of the equations:

a. $4x - (x + 2) = 10$

b. $8r - (6r + 5) = 7$

c. $5x - (2x - 6) = 20$

d. $3y - (4 - 2y) = 21$

e. $x - (.4 - x) = 1.6$

f. $(6x + 3) - (3x + 5) = 13$

g. $(2a + 6) - (a - 7) = 10$

h. $(\frac{5}{2}x - 2) - (\frac{3}{2}x + 5) = 1$

CHAPTER VI

EQUATIONS AND INEQUALITIES

In this chapter are contained suggested procedures for helping pupils develop understanding and facility in solving equations which have the variable in both members, and in applying this skill to the solution of verbal problems. Here, too, are presented suggestions for developing pupil understanding and ability in the solution of literal equations and simple inequalities.

I. Equations with Variables on Both Sides of the Equal Sign

A. Suggested Procedure

1. Have pupils solve equations with variables on both sides of the equals sign by trial and error.

   a. Pose problem:

   If three times a number is increased by four, the result is the same as the original number increased by sixteen.

   b. Have pupils translate the English sentence into a mathematical sentence. If n represents the number, $3n + 4 = n + 16$.

   c. The pupils should realize that $3n + 4 = n + 16$ (the domain of n is the set of signed numbers) is an open sentence.

   Review meaning of open sentence. Have them notice that in this case the variable appears in both the left and right members of the equation. Have them select numbers from the domain and try to discover the value of n which makes the sentence true (if such a value exists).

   d. Have the pupils try to solve other equations with variables on both sides by trying to discover the value of the variable which makes the sentences true.

2. Have pupils realize that a more systematic procedure is needed to solve equations having the variable in both members.

   a. Have pupils recall the use of the additive and multiplicative inverses in transforming an equation such as $3x + 5 = 17$ to simpler equivalent equations.

$$3x + 5 = 17$$

$$3x + 5 + (-5) = 17 + (-5)$$

$$3x = 12$$

$$\frac{1}{3}(3x) = \frac{1}{3}(12)$$

$$x = 4 \qquad 4 \text{ is a root of } x = 4$$

Check to show that 4 is also a root of $3x + 5 = 17$

b. Have pupils try to discover whether the principles used in 2a will be helpful in obtaining simpler equations equivalent to $3n + 4 = n + 16$.

c. The discoveries of the pupils should lead to the following form:

$$3n + 4 = n + 16$$

$3n + 4 + (-4) = n + 16 + (-4)$ Addition principle

$3n = n + 12$ Why?

$3n + (-n) = n + 12 + (-n)$ Addition principle

$2n = 12$ Why?

$\frac{1}{2}(2n) = \frac{1}{2}(12)$ Multiplication principle

$n = 6$ Why?

Have pupils check to see that 6 is a root of the equation $3n + 4 = n + 16$ and that 6 satisfies the conditions of the given problem.

Have pupils realize that it would be perfectly correct to add $-n$ to each member as the first step in obtaining an equivalent equation with the variable in only one member.

B. Suggested Practice

Solve each of the following equations:

1. $9x = 3x + 18$
2. $3a + 18 = 6a$
3. $3n + 6 = 2n + 10$
4. $2 + 5b = 4b + 8$
5. $2y + .5 = 2.5 + y$
6. $.1 + 7m = .25 + 6m$
7. $2x + \frac{3}{4} = 1 + x$
8. $3x - 4 = 11 - 2x$
9. $3x + 5 = 7x + 33$
10. $2(2y - 4) = y$
11. $4x = 2(x - 4)$
12. $27 - 4b = 15 + 3b$

13. $2(x + 3) = 3(x - 2)$

14. $5r - .3 = 1.8 - 2r$

15. $-25x = 90 + 5x$

16. $1 + 3x = x + \frac{1}{2}$

17. $x = 2(90 - x)$

18. $3(3x + 1) = 4(2x + 2)$

19. $\frac{1}{5}(y - 10) + 16 = \frac{3}{5}y + 12$

20. $2x - 4(x + 2) = 5(x + 4)$

__Note:__ 1. When pupils show understanding of the "proof" in 2-c, the arrangement of work may be condensed as follows:

$$3n + 4 = n + 16$$

$$3n = n + 12$$

$$2n = 12$$

$$n = 6$$

Check: $3(6) + 4 \overset{?}{=} (6) + 16$

$22 = 22$ True

2. When using condensed form for solving an eqatuion, occasionally have pupils state how they arrive at each equivalent equation and the justification therefor.

II. Verbal Problems

A. Suggested Procedure

1. __Solve__: Nine more than twice a number is the same as six times the number decreased by three.

__Solution__: Let x = the number

then, $2x + 9$ = nine more than twice the number

and $6x - 3$ = six times the number decreased by 3

then, $2x + 9 = 6x - 3$

$2x + 12 = 6x$

$12 = 4x$

$3 = x$ The number is 3.

__Check__ results with __conditions__ in the __problem__.

If the number is 3, then nine more than twice the number is $2 \times 3 + 9$ or 15. This is the same as six times the number decreased by three, that is, $6 \times 3 - 3$ or 15.

2. **Solve**: A dealer has 10 radios on sale. Some of these radios were sold for \$25 each and the remainder for \$20 each. The total sales value of the \$25 radios exceeded the total sales value of the \$20 radios by \$160. How many radios were sold at each price?

Solution: Let n = number of radios on sale at \$25

then $10-n$ = number of radios on sale at \$20

$25n$ = sales value of all radios sold at \$25 each

$20(10 - n)$ = sales value of all radios sold at \$20 each

then $25n = 20\ (10 - n) + 160$ (equation)

$25n = 200 - 20n + 160$

$45n = 360$

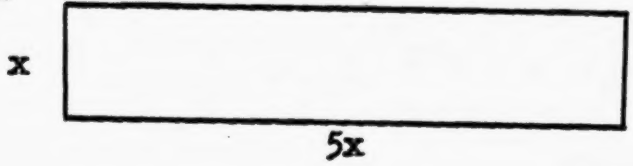
$n = 8$ 8 radios sold at \$25 each

$10 - n = 2$ 2 radios sold at \$20 each

Check results with conditions of the problem.

If 8 radios sold at \$25 each, their total sales value is \$200.
If 2 radios sold at \$20 each, their total sales value is \$40.
It is true that \$200 exceeds \$40 by \$160.

3. In the figure at the right, the length of the rectangle is five times the width.



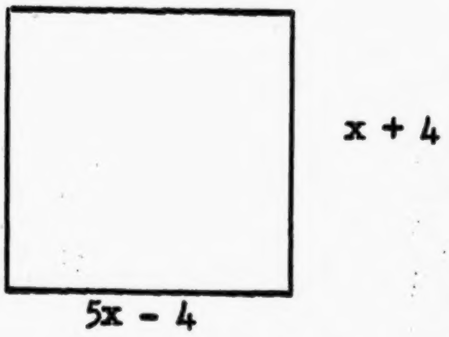
If the length is decreased by 4, and the width increased by an equal amount, the figure becomes a square. Find the dimensions of the rectangle.

Solution: Let x = width of the rectangle

then, $5x$ = length of the rectangle

and,
$x + 4$ = the length of one side of the square

$5x - 4$ = the length of an adjacent side of the square



Since all sides of a square are equal, we write

$x + 4 = 5x - 4$
Then, $8 = 4x$ Why?
$2 = x$
$10 = 5x$

Dimensions of the rectangle are:
Width = 2; Length = 10

Check, using the conditions stated in the problem.

The length, 10, of the rectangle is 5 times the width, 2.

If the length is decreased by 4, we have 10 - 4 or 6.

If the width is increased by 4, we have 2 + 4 or 6.

The resulting figure is a square.

4. Solve: A freight train starts out at 12 noon and is followed at 1 p.m. by a passenger train which goes 6 mph faster. If the passenger train overtakes the freight train at 5 p.m. find the rates of the two trains.

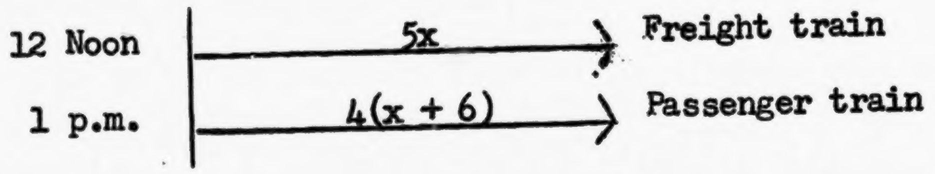
Solution: Let x = rate of freight train in mph

$x + 6$ = rate of passenger train in mph

Have pupils organize the facts in table form:

	Rate (in mph) x	Time (in hours) =	Distance (in miles)
Freight train	x	5	$5x$
Passenger train	$x + 6$	4	$4(x + 6)$

Have pupils draw a distance diagram:



Since each train covered the same distance, we have

$5x = 4(x + 6)$

$5x = 4x + 24$ (Why?)

$x = 24$ (Why?) The rate of the freight train is 24 mph.

$x + 6 = 30$ The rate of the passenger train is 30 mph.

Check results with conditions in the problem.

B. Suggested Practice

1. If I subtract 15 from six times a certain number, the result will be three times the number. Find the number.

2. Four times a certain number when increased by three is equal to three times the number increased by five. Find the number.

3. John is now twice as old as Peggy. Ten years ago, John was four times as old as Peggy. What is the present age of each person?

4. Michael is 2 years older than Barry. Twelve years ago Michael was twice as old as Barry. Find their ages.

5. Train A leaves a station at a certain speed and travels for three hours. Train B leaves the same station one hour later, traveling at a rate 20 mph faster but in the opposite direction, for two hours. Upon arrival at their respective destinations, the trains have traveled the same distance. Find the rate at which each train has traveled.

6. Car C arrives at a destination after traveling for four hours at a constant speed. Car D leaves two hours later, traveling at a rate 20 mph faster than Car C. How fast was each car traveling, if, at the end of two hours, Car D is still 6 miles from the destination.

7. Two angles whose sum is 90° are said to be complementary. If one of the two complementary angles is 30° more than twice the other, how many degrees are there in each angle?

8. George invested $500 in two parts. One part is invested at 6% and the second part at 2%. If the interest on the part invested at 6% exceeds the interest on the part invested at 2% by $14, find how much money was invested at each rate(interest per year).

9. A man invested $4,200, part at 5% and the remainder at 4%. The annual interest on the money invested at 5% is $24 less than the annual interest earned on the money invested at 4%. Find how much money the man invested at each rate.

10. Mr. Jones made a mixture of 100 pounds of candies. Some of the candy was worth 80¢ per pound and some was worth 60¢ per pound. The value of the 80¢ candy in the mixture exceeds the value of the 60¢ candy in the mixture by $45. How many pounds of each kind of candy went into the mixture?

11. If, in a square, one side is increased by 16, the new figure has a length 8 inches greater than twice the side of the original square. Find the length of a side of the original square.

12. If, in a square, one side is increased by 6 and the adjacent side decreased by 3, the newly formed rectangle has a length which is 4 more than twice its width. Find the length of a side of the original square.

13. A dealer blends 100 pounds of coffee using coffees worth 80¢ and 40¢ per pound. If the total value of the 40¢ coffee is twice the total value of the 80¢ coffee, how many pounds of each kind of coffee were blended in the mixture?

III. Formulas

Note: Equations containing more than one variable occur frequently in algebra as well as in the application of mathematics to science and business.

A. Suggested Procedure for Formulas Having Two Variables

1. If the equipment can be obtained, the teacher can have the pupils assist in performing the following simple experiment:

   Attach a series of graduated weights to a spring and measure and record the corresponding stretches in the spring. Thus, when there is no weight attached to it, the spring reaches a point that we let correspond to zero. When a weight is attached, the spring stretches, and the distance it stretches may be measured using an appropriate vertical scale. A table such as the following can be constructed:

W (weight in pounds or grams)	0	1	2	3	4	5
S (stretch in inches or centimeters)	0	2	4	6	8	10

   Have pupils examine this data in search of a pattern. They should be able to reach the generalization that the number of inches (S) that the spring stretches is equal to twice the number of pounds (W) in the weight attached to the spring. (Since different springs and weights may be used, the coefficient may be different. However, the relationship will always be of the first degree.) Have pupils express this relationship mathematically as: $S = 2W$

2. Pose question: How does this equation differ from those which have been studied previously?

   Elicit: This equation has two variables, one of which is expressed in terms of the other.

3. Have pupils suggest at random possible values for S and W. Substitute these pairs of values to determine whether the statement which results is true or false. Since in some cases a false statement will result, pupils conclude that $S = 2W$ is like an open sentence in that it is true for certain selections of S and W and not true for others. Thus, $S = 2W$ is true if W is 3 and S is 6, but false if W is 3 and S is 8.

4. Discuss the solution set of $S = 2W$. Have pupils observe that the solution set consists of elements which are pairs of numbers such that the value for S is always twice the value for W.

   For example, (1,2), (2,4), (4,8) are elements of the solution set where the first number of the pair indicates a number of pounds and the second number indicates a number of inches.

5. Discuss the domain of W from the standpoint of meaningful replacements for W. Guide pupils to see that it is reasonable to limit the replacement set for W to non-negative numbers (positive numbers and zero). Suppose, too, the spring cannot withstand a weight of more than 10 pounds. Then it would be meaningless to state that the number pair (11,22) is an element of the solution set.

6. Have pupils consider that sometimes we would like to know what weight will produce a certain stretch. What weight must be used to obtain a stretch of 2 inches, 3 inches, 5 inches, 8 inches? Have them realize that we divide the given value of S by 2 to obtain the corresponding value for W.

   That is, $W = \frac{S}{2}$

   Guide pupils to see that in $S = 2W$, S is expressed in terms of W, but in $W = \frac{S}{2}$, W is expressed in terms of S. The latter arrangement of the formula is called the solution of the equation $S = 2W$ for W in terms of S.

   Solving an equation for one variable in terms of the other is a convenient way of determining the value of that variable when values of the other are given.

7. Have pupils realize that since formulas are equations, the rules for solving equations may be applied to the solution of formulas. In applying these rules, it may be helpful at first to have pupils replace the variables other than the one to be solved for by any numerals other than 0. The form of the equation then becomes recognizable. Example: Solve $p = 4s$ for s.

   If p is replaced by 8, we have $8 = 4s$ which is solved by using the division (or multiplication by reciprocal) principle.

Then dividing both members of $p = 4s$ by 4: $\frac{p}{4} = s$.

B. Suggested Practice

Solve for the variable which is underlined.

1. $d = 2\underline{r}$ (Diameter of a circle)
2. $I = .06\ \underline{P}$ (Interest for 1 year at 6% on P dollars)
3. $F = 6\underline{a}$ (Force = mass times acceleration where the mass is 6)
4. $P\underline{V} = 6$ (Boyle's Law. P and V must be positive)

C. Suggested Procedure for Formulas with More Than 2 Variables

1. Follow the procedure suggested in III-A and discuss the meaning of the variables whose possible values have limitations. For example,

   a. $I = P\ R\ T$

      R - often has statutory limitations

      P - discuss F.D.I.C. limitation of $15,000 on deposits

      T - pension systems limit number of years for which interest is paid to one who has resigned

   b. $\frac{PV}{T} = .08$ (Charles Law)

      T, temperature, must be positive (measured on the absolute scale)

      P, pressure, must be positive

      V, volume, must be positive

2. Solve $p = 2\ell + 2w$ for w.

   If we replace the other letters (p and ℓ) by any numerals other than 0, say by 24 and 3, we have $24 = 6 + 2w$ (a form recognizable by the pupils).

   Subtracting 2 (or adding the additive inverse of 2) from both members: $p - 2\ell = 2w$.

   Dividing both members by 2 (or multiplying by the reciprocal of 2): $\frac{p - 2\ell}{2} = w$

3. Have pupils solve in similar fashion for the underlined variable in equations such as:

1) $A = \underline{\ell}w$	(Area of a rectangle)
2) $p = 2\underline{a} + b$	(Perimeter of an isosceles triangle)
3) $V = \frac{1}{3}\underline{B}h$	(Volume of a cone)
4) $E = R\,\underline{I}$	(Ohm's Law)

Note: When dividing by a variable, zero must be excluded from its domain.

D. Suggested Practice

Solve for underlined variable. Discuss the meaning of each of the variables.

1. $A = \underline{b}h$
2. $d = r\underline{t}$
3. $I = P\,\underline{R}\,T$
4. $p = \underline{a} + b + c$
5. $p = 2\underline{\ell} + 2w$
6. $A = \frac{1}{2}b\underline{h}$
7. $V = \ell\underline{w}h$
8. $V = \pi r^2\underline{h}$
9. $s = \frac{1}{2}\underline{a}t^2$
10. $H = \underline{v}t - 16t^2$

IV. Inequalities

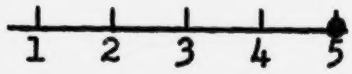
A. Suggested Procedure

1. Review meaning of statements of inequalities; symbols $\neq, <, >$; open sentences expressing inequalities; order on a number line, (See Mathematics Grade 8, Curriculum Bulletin #4, Series 1961-62, pages 62-64, 130-140, 156-157).

2. Review graph of equation in one variable on a number line.

   a. Have pupils solve: $x + 7 = 12$ where the domain of x is $\{1, 2, 3, 4, 5\}$. Solution set of $x + 7 = 12$ is 5 .

   b. The graph of the solution set is the point corresponding to its only member, 5.

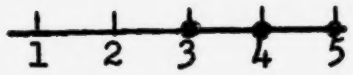


   Note: The graph of the solution set is also called the graph of the equation.

3. Have pupils graph inequalities in one variable on a number line.

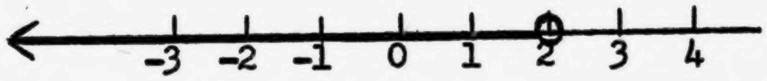
   a. Have them solve: $x > 2$ where the domain of x is $\{1, 2, 3, 4, 5\}$. Solution set is $\{3, 4, 5\}$.

b. The graph of the inequality is the graph of its solution set. The graph of $x > 2$ for the indicated domain appears at the right.



c. Solve and graph: $x < 2$ when the domain is {signed numbers}; the pupils should realize that 2 is not in the solution set. The solution set is {all signed numbers less than 2}.

The graph of this solution set is



This is also called the graph of the inequality $x < 2$.

Note: 1. The "open circle" at 2 indicates that 2 is not a member of the solution set.

2. The heavy arrow extending to the left indicates the graph extends indefinitely in that direction.

d. By testing various elements of the replacement set for x, have pupils solve and then graph several inequalities such as:

1) $x > 5$

2) $x > 0$

3) $x + 2 > 4$

4) $x + 1 < -3$

The domain of x in each case is the set of signed numbers.

4. Have pupils learn the meaning of the symbols: "$\leq$" and "$\geq$"

a. Pose problem:

To vote in New York State, one condition a citizen must meet is that he or she must be 21 years or older. How would you represent this condition with the use of algebraic symbols?

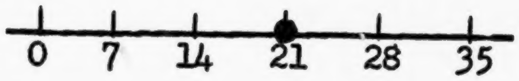
If a represents the number of years in the age, then

1) $a = 21$
or
2) $a > 21$

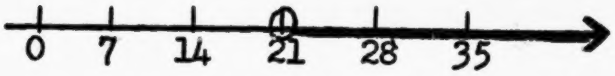
Why are the two sentences needed?

b. If the two sentences, $a = 21$ or $a > 21$ are graphed on the same number line, the condition for voting may be shown graphically:

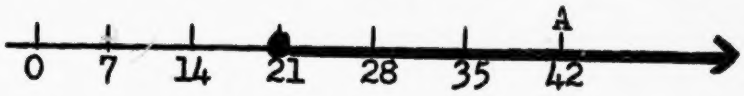
1) Sentence 1) $a = 21$



2) Sentence 2) $a > 21$



3) Combining 1) and 2)



Then the graph of $\underline{a = 21 \text{ or } a > 21}$ consists of the point which corresponds to 21 and also all points to the right.

4) Have pupil select any number associated with any point of the graph and check to see if this number satisfies the condition for voting. For example, point A has the number 42 associated with it. 42 satisfies the condition.

c. Tell pupils that the sentence $a = 21$ or $a > 21$ is conventionally written $a \geq 21$ (read "a is greater than or equal to 21").

d. Similarly, the symbol $\leq$ can be used to express the following statement: The speed limit on the New York State Thruway is 60 mph.

If $s =$ the speed of a car in mph

1) $s = 60$ is legal
2) $s < 60$ is legal
3) Statements 1) and 2) can be combined $s \leq 60$.

Have pupils graph sentences 1), 2) and 3).

e. Suggested Practice

1) Express each of the following as a mathematical sentence:

   a) n is greater than or equal to 5.
   b) y is less than or equal to 2.

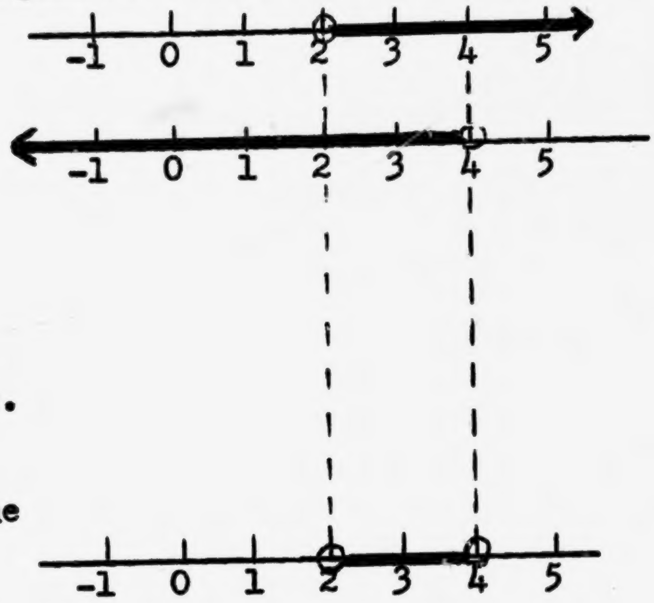
2) Suppose a represents any number in the set of integers. Name two replacements for a that will make the sentence $a \leq 6$ true.

   Name one replacement that will make it false.

3) What is the meaning of $m \leq 35$, where m is the greatest number of pupils permitted in the mathematics club?

4) Graph the components of $m \leq 35$ ($m = 35$ or $m < 35$), then draw the graph of $m \leq 35$.

5. The meaning of combined inequalities such as $2 < x < 4$.



a. The graph of $2 < x$ is

b. The graph of $x < 4$ is

c. Have pupils suggest numbers which satisfy both inequalities, for example, 3, $2\frac{1}{2}$, $2\frac{1}{4}$, etc.

Have them locate these numbers on a number line where x satisfies both conditions.

Then the graph of $2 < x$ and $x < 4$ consists of all the points between the points which correspond to 2 and 4. The points corresponding to 2 and 4 are excluded.

d. Have pupils see that $2 < x$ and $x < 4$ may be written as $2 < x < 4$; and may be read "x is greater than 2 and less than 4."

e. Suggested Practice

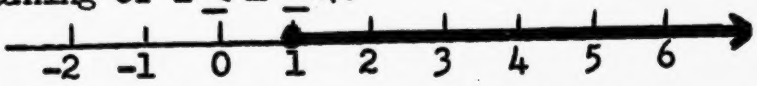
1) Combine $3 < m$ and $m < 8$ into one expression.

2) What is the meaning of $-5 < y < 2$? Graph the two parts of $-5 < y < 2$. Graph $-5 < y < 2$.

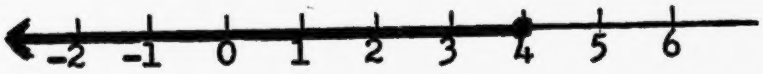
3) Draw the graph of $0 < m < 6$. Discuss the values of m that satisfy this inequality.

f. Have pupils consider the meaning of $1 \leq m \leq 4$.

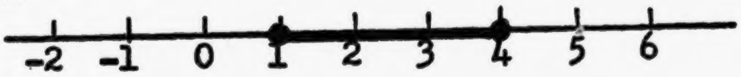
1) The graph of $1 \leq m$ is



2) The graph of $m \leq 4$ is



3) The graph of $1 \leq m \leq 4$ is



It consists of the points which correspond to 1 and 4 and also all the points between these points.

g. Suggested Practice

1) What is the meaning of $-3 \leq m \leq 0$? Draw the graph of $-3 \leq m \leq 0$.

2) What is the meaning of $2 \leq j \leq 9$? Draw its graph.

3) Which two sentences form the single sentence: $-4 < k \leq 4$?
Draw the graphs of the individual sentences.
Draw the graph of $-4 < k \leq 4$.

4) Write in mathematical symbols:

b is greater than or equal to a and less than c.

5) OPTIONAL

What is the meaning of each of the following inequalities?

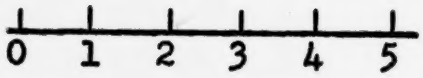
a) $|x| \leq +5$ (means $-5 \leq x \leq +5$)

b) $|x| \geq +5$ (means $x \geq +5$ or $x \leq -5$)

Draw the graphs of each of the above.

6. Inequalities related to equations

a. Review meaning of inequality by reference to order on the number line.



$4 > 1$ because 4 is to the right of 1 on the number line.

$4 > 1$ means the same as $1 < 4$.

b. Have pupils consider $1 < 4$. Which is the smaller number? (1) What must be added to the smaller number to make it equal to the larger number? (3)

Therefore, $1 + 3 = 4$.

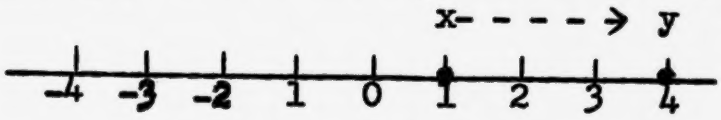
In $0 < 4$, what must be added to 0 to make it equal to 4?

In $-3 < -2$, what must be added to -3 to make it equal to -2?

Do you think it is always possible to find a number that can be added to the smaller number to yield a number that will be equal to the larger number?

What can you say about the sign of that number that is added to the smaller number in each case? (It is positive.)

c. Have pupils consider the inequality x y. If the numbers x and y are represented by points on the number line, then the point representing y lies to the right of the one representing x.



How can we get from x to y? (We move to the right.) This means that y can be obtained by adding some positive number to x, since a move to the right on the line corresponds to the addition of a positive number.

Have pupils realize that this gives us a way of defining the relation "greater than" and "less than" in terms of an equation.

1) If $x < y$, there is a positive number p such that $x + p = y$.

2) If $x > y$, there is a positive number p such that $x = y + p$.

These are summarized: There is a positive number, p, which when added to the smaller number will make it equal the larger number.

d. Suggested Practice

1) For each pair of numbers, determine their order and find the positive number p which when added to the smaller gives the larger:

a) -15 and -24

b) $1\frac{3}{4}$ and $\frac{3}{4}$

c) $\frac{4}{5}$ and $\frac{7}{10}$

d) $-.3$ and $.05$

2) Use $a + b = c$ where a and b are positive numbers to suggest two true sentences involving " $<$ ", relating pairs of the numbers. ($a < c$; $b < c$).

Can we conclude that a   b?

7. Principles (postulates) for operating on inequalities

a. Review with pupils the meaning of equivalent equations, i.e., two equations are called equivalent if they have the same solution set.

Thus, $2x + 5 = 17$ and $x = 6$ are equivalent equations. The latter equation is one whose solution is obvious.

b. Have pupils understand the meaning of equivalent inequalities.

1) Consider the inequality $x + 5 < 9$. What is its solution set? Certainly, 4 cannot be a solution, since 4 + 5 equals 9. Nor can any number greater than 4 be a solution, since the sum of that number and 5 would be greater than 9.

However, every number less than 4 is a solution, since the sum of such a number and 5 is bound to be less than 9. For example, statements such as the following would all be true:

$1 + 5 < 9$ $3\frac{1}{2} + 5 < 9$ $-10 + 5 < 9$

Then the solution set of $x + 5 < 9$ is $\{\text{all numbers} < 4\}$.

2) Consider the inequality $x < 4$. It is obvious that its solution set is $\{\text{all numbers} < 4\}$.

3) Inequalities such as $x + 5 < 9$ and $x < 4$ are called equivalent inequalities because they have the same solution set.

c. Recall with pupils the principles used in solving equations. Will similar principles hold true for inequalities? Since, as we have seen, inequalities can be related to equations, there is good reason to explore the possibility that similar principles will hold true.

Let us experiment to discover which, if any, will preserve the order of inequality and result in equivalent inequalities:

1) Consider the inequality $6 < 18$
Adding 5 to each number $6 + 5 < 18 + 5$ true

Again, $6 < 18$
subtracting 5 (adding -5) $6 - 5 < 18 - 5$ true

These two cases can be illustrated by two men walking along a number line, carrying a 12 unit ladder, one man at each end. If they move 5 units in either direction, they will still be 12 units apart. The man originally to the right will still be to the right. The order or the sense of the inequality is preserved.

It appears then that: If the same number is added to or subtracted from both members of an inequality, the resulting inequality has the same order as the original.

2) Now examine the inequalities: $x + 5 < 9$ and $x < 4$

We have already seen these are equivalent. If we add -5 to both sides of the first, we get the second. After examining many such illustrations, it appears that the following principle holds for inequalities:

Adding (or subtracting) any number (or polynomial) to both sides of an inequality results in an equivalent inequality, with the order preserved. This is called the Addition Principle of Inequalities.

3) Consider the inequality $6 < 18$

Multiplying by 2, $12 < 36$ true

Again, $6 < 18$

Multiplying by -2, $-12 \ ? -36$

Since -12 is greater than -36 $-12 > -36$

In the last illustration, what has happened to the order of the original inequality? Is this an accident? Try some more.

It seems reasonable to conclude that:

a) If both members of an inequality are multiplied by the same positive number, the resulting inequality has the same order as the original.

b) If both members of an inequality are multiplied by the same negative number, the order is reversed.

Similar experimentation with division leads to the conclusion that:

c) If both members of an inequality are divided by the same positive number, the resulting inequality has the same order as the original.

d) If both members of an inequality are divided by the same negative number, the order will be reversed.

e) What happens when both sides of an inequality are multiplied by 0? (The inequality becomes an equality: $2 < 7$; $2 \times 0 = 7 \times 0$)

4) Consider the inequalities: $3x < 9$ and $x < 3$

By procedures similar to 7-b 1)-3), these can be shown to be equivalent. If we multiply each side of the first inequality by $\frac{1}{3}$, we get the second. Examination of many such illustrations makes it appear that the following principle holds for inequalities:

Multiplying (or dividing) both sides of an inequality by the same positive number results in an equivalent inequality with the order preserved.

Multiplying(or dividing) both sides of an inequality by the same negative number results in an equivalent inequality with the order reversed.

This is called the Multiplication Principle of Inequalities.

8. (OPTIONAL)

The approach used in 7 is designed to have pupils suggest propositions which appear to be reasonable. These propositions can be proved as illustrated below.

This proof is recommended for enrichment only.

Proposition: If $x < y$, then for any signed number b, $x + b < y + b$.

Proof:	
1) $x < y$	1) Given
2) $x + n = y$	2) Definition of x y, where n is positive
3) $x + n + b = y + b$	3) Addition principle for equations
4) $(x + b) + n = y + b$	4) Associative principle, Commutative principle
5) $x + b < y + b$	5) Definition of Inequality

9. Solution of open sentences involving inequalities

a. Have pupils give the solution of inequalities in one variable when the variable alone is on one side of the inequality sign. For example, (the domain is the set of signed numbers in each case)

$n < 3$ $\qquad$ $x > 4$ $\qquad$ $3 < y$ $\qquad$ $-1 > n$

They will realize the solutions are obvious in the case where the variable alone is on one side of the inequality sign.

b. Have them use the principles to solve inequalities such as

1) $y + 2 > 5$

$y + 2 + (-2) > 5 + (-2)$ Why?

$y > 3$ The solution set is $\{$all numbers $> 3\}$.

2) $-3x \leq 15$

$-\frac{1}{3}(-3x) \geq -\frac{1}{3}(15)$ Why?

$x \geq -5$ The solution set is $\{$ all numbers $\geq -5\}$.

B. Suggested Practice

1. Solve and graph

a. $b - 3 > 7$ $\qquad$ b. $d + 4 < 15$ $\qquad$ c. $m + 2 < 0$

d. $8 < d - 6$ $\qquad$ e. $5 > 1 - y$ $\qquad$ f. $4b > 8$

g. $3d \leq 9$ h. $-4y > 8$ i. $-2b \geq -6$

j. $-2b \geq -6$ k. $-4y < -8$ l. $.1r \geq -6$

m. $4a - 1 > 1$ n. $2r < r + 4$ o. $5 + \frac{x}{3} \geq 0$

p. $\frac{1}{2}y - 2 < -8$ q. $3b - 4 < b - 2$ r. $8 - 2y > 16 - 6y$

2. Verbal expressions involving inequalities. Use mathematical symbols to indicate these quantitative situations. Indicate what the variable represents.

   a. Applicants for a certain job must be (a) more than 21 years of age; (b) at least 21 years of age.

      (a) $m > 21$ (b) $m \geq 21$ where m = number of years in age.

   b. The speed limits on this highway are: minimum 30 mph and maximum 60 mph ($30 \leq s \leq 60$)

   c. It takes me from 15 minutes to $\frac{3}{4}$ hour (exclusive) to do my homework.
      ($15 < m < 45$ or $\frac{1}{4} < t < \frac{3}{4}$)

   d. The sum of two sides of a triangle is greater than the third side. If two sides are 10 and 15, write an inequality describing the third side.
      ($m < 10 + 15$)

3. Verbal problems involving inequalities

   a. Assume 120 pounds as an average weight per pupil. If we are told that the total weight of pupils in the elevator of a building at one time must be less than 2400 pounds, what is the largest number of pupils permitted in the elevator at one time?

      Solution: Let n = the largest number
      Then
      $120\,n < 2400$
      $n < 20$

      The domain of the variable is the set of positive integers. Since n must be an integer, the answer is 19 pupils.

   b. The post office requires that the length and girth of a package, under certain conditions, must not exceed 72 inches. If the girth is 40", find the number of inches that the length may be.

Solution: Let n = the number of inches in the length

$$40 + n \leq 72 \text{ or}$$

$$n \leq 32$$

The length may be any positive number less than or equal to 32 inches, since the problem implies that the replacement set for the variable must be a positive number.

Note: The solution set may have an infinite number of elements as in example b, or it may have one element as in example a, or it may conceivably have two or more, or even no elements.

c. Betty and Alice were collecting money in a fund-raising campaign. At present they have collected less than $15. Alice said she had collected twice as much as Betty. How much had Betty collected?

Solution: Let b = the number of collars Betty had collected
Then 2b = the number of dollars Alice had collected

$$3b < 15$$

$$b < 5$$

Betty collected an amount less than $5. We cannot tell how much Betty collected until we have more information. We can only state one fact, that it is an amount less than $5. If the question were, "What is the largest amount Betty might have collected?" we could definitely say $4.99.

d. For enrichment only:

A trip will be arranged for a minimum of 10 pupils in a group and a maximum of 30 pupils in a group.

How many groups might have to be scheduled when 65 pupils apply?

Some pupils may discover the solution: Any number between 2 and 7, where the domain of the variable is the set of positive integers $2 < x < 7$.

CHAPTER VII

SYSTEMS OF EQUATIONS AND GRAPHS

This section contains suggested procedures for the teacher to help pupils develop the following understandings and skills:

1. Graphing a linear equation
2. Graphing an inequality
3. Solving systems of equations by means of a graph
4. The algebraic solution of systems of equations
5. Application to problems
6. Graphing systems of inequalities

I. Open Sentence in Two Variables

A. Suggested Procedure

1. Have pupils learn that solutions of sentences in two variables are number pairs.

Pose problem: Pupils of the Longview High School sold tickets for a music festival and collected \$50 in the first two days. Pupil tickets were sold at \$1 each, and adult tickets at \$2 each. How many of each kind were sold?

Guide pupils to describe the above problem as an open sentence in two variables as follows:

If x = number of pupil tickets
y = number of adult tickets
then $x + 2y = 50$

What are some possible values of x? $\{0, 1, 2, \ldots, 50\}$

What are some possible values of y? $\{0, 1, 2, \ldots, 25\}$

Have pupils try various replacements for x and y.

x	y	$x + 2y = 50$	
4	23	$4 + 2(23) = 50$	true statement
6	22	$6 + 2(22) = 50$	true statement
22	6	$22 + 2(6) = 50$	false statement
10	20	$10 + 2(20) = 50$	true statement

a. Pupils will realize that whereas a solution of an open sentence in one variable is a number, a solution of an open sentence in two variables is a number pair. Thus, some possible solutions to the problem are 4 pupil tickets and 23 adult tickets, or 6 pupil tickets and 22 adult tickets, or 10 pupil tickets and 20 adult tickets.

b. Tell pupils that number pairs such as 4 (the replacement for x) and 23 (the replacement for y) are conventionally written $(4, 23)$.

c. Have pupils see that if we agree that the value of x is to be given first, then the pair $(4, 23)$ makes the sentence $x + 2y = 50$ true, but the pair $(23, 4)$ makes it false.

d. Tell pupils that a pair of numbers in which the order is important is called an ordered number pair.

e. An ordered pair of numbers which makes an open sentence true is called a solution of the open sentence. The solution set of an open sentence in two variables is the set of all solutions of the sentence.

2. After several problems in which pupils find the solution set of an open sentence in two variables by trying various replacements for each variable, guide them to realize that this is time-consuming and that a more efficient method is needed. Help pupils find the solution set of an open sentence in two variables as follows:

Find the solution set of $2x + y = 8$ when the replacement set for both x and y is $\{0, 1, 2, 3, 4\}$.

a. Change $2x + y = 8$ into an equivalent equation with y alone as one member:

$y = 8 - 2x$ (Addition principle of equations)

b. Replace x by each element of its replacement set, in turn, and determine the corresponding value of y.

x	$8 - 2x$	y
0	$8 - 2(0)$	8
1	$8 - 2(1)$	6
2	$8 - 2(2)$	4
3	$8 - 2(3)$	2
4	$8 - 2(4)$	0

c. Observe which values of y in the above table are members of the replacement set for y. These are 0, 2, 4. Then the ordered pairs (2,4), (3,2), (4,0) are the solutions of the open sentence $2x + y = 8$. The solution set is $\{(2,4), (3,2), (4,0)\}$.

d. Have pupils realize the solution set may contain an unlimited number of number pairs. For example, when the replacements for x and y are the signed numbers, the solution set of the equation $2x + y = 8$ contains an unlimited number of ordered number pairs.

Note: If the open sentence contains variables other than x and y as, for example, a and b, the replacement for a is usually given first in the ordered number pair representing a solution. That is to say, the first number of the ordered pair is the replacement for the variable which is first in alphabetical order.

3. Have pupils find the solution set of an inequality in two variables as follows:

a. Pose problem: The pupils in Miss Andrew's class decided to sell boxes of candy to raise money for some library equipment. Each pupil was asked to take as many boxes as he thought he could sell. Bob took 2 boxes, and Harry took 6. If, together, they sold more than 5 boxes, how many did each sell?

Guide pupils to describe this problem as an open sentence in two variables as follows:

If x = number of candy boxes Bob sold
y = number of candy boxes Harry sold
then $x + y > 5$

Elicit that the replacement set for x is $\{0, 1, 2\}$ and the replacement set for y is $\{0, 1, 2, 3, 4, 5, 6\}$.

b. Have pupils change $x + y > 5$ into an equivalent inequality with y alone as one member:

$y > 5 - x$ (Addition principle of inequality)

c. Have pupils replace x by each element of its replacement set, in turn, and determine the corresponding value of y.

x	$5 - x$	$y > 5 - x$	y
0	$5 - 0$	$y > 5$	6
1	$5 - 1$	$y > 4$	5,6
2	$5 - 2$	$y > 3$	4,5,6

d. Have pupils see that the solutions of the open sentence are the ordered number pairs:

$(0,6), (1,5), (1,6), (2,4), (2,5), (2,6)$

and the solution set is

$\{(0,6), (1,5), (1,6), (2,4), (2,5), (2,6)\}$

e. Have pupils check each element of the solution set against the problem situation.

Note: An equation (or inequality) containing three variables has a solution set consisting of ordered triples, etc. Whereas there is no limit to the number of variables an equation (or inequality) may have, the work of this course will, in general, be confined to open sentences with no more than two variables.

B. Suggested Practice

1. If the replacement set of each variable is the set of signed numbers, tell whether the given ordered pair of numbers is a solution of the open sentence.

$x + y = 3$;	$(2,1)$	Yes, because $2 + 1 = 3$
$3x + y = 6$;	$(1,3)$	
$a + 4b = 9$;	$(3,2)$	No, because $3 + 4(2) \neq 9$
$2x - 2y = 10$;	$(2,-3)$	
$2m + 3n = 2$;	$(\frac{1}{2},\frac{1}{3})$	
$2x - y > 2$;	$(2,1)$	
$a + 3b < 7$;	$(1,3)$	

2. Change each open sentence into an equivalent one which has y alone as one member; x alone as one member.

$3x + y = 8$

$x - y = 12$

$5x - y = 0$

$4y = 8x - 4$

$x - y = -5$

$x + y > 3$

$6x + 3y > 9$

$y - x < 5$

3. Find the solution set of each of the following sentences. The replacement set for x is $\{-3, 0, 3\}$ and the replacement set for y is $\{\text{signed numbers}\}$.

$y = x + 4$
$y = 3 - x$
$2x + y = 6$
$y + 5x = 9$
$y - 4x = 8$
$5x - 2y = 10$
$2x + 3y = 24$

4. Find the solution set. In each case the replacement set for x is indicated first, and then the replacement set for y.

$x - y = 7$	$\{-4\}$	$\{\text{integers}\}$
$x + 3y = 9$	$\{0\}$	$\{\text{integers}\}$
$6 - 3y = 2x$	$\{0, 9\}$	$\{\text{signed numbers}\}$
$x + 4y = 10 - x$	$\{4, 8\}$	$\{\text{signed numbers}\}$
$2 - x = 5y + 3x$	$\{0, 5\}$	$\{\text{signed numbers}\}$
$y + 1 > 3x$	$\{0, 1\}$	$\{0, 4, 5\}$
$x + y > 9$	$\{2, 4, 6\}$	$\{3, 5, 7\}$
$y < 2x + 1$	$\{-2, -1, 0\}$	$\{-8, -5, 0\}$

5. Describe each of the following word problems as an open sentence with two variables. Give replacement sets and solution sets for each.

   a. If peanuts cost 50¢ per pound and cashews cost 75¢ per pound, how many pounds of each can be purchased for $5?

   b. A storekeeper has $200 to spend for sweaters and blouses. If sweaters cost $4 each and blouses cost $2 each, how many of each can he buy?

   c. I am thinking of two positive integers. If the second number is added to twice the first number, the result is always less than 11. Find the numbers.

II. Coordinates in a Plane

A. Suggested Procedure

1. Review with pupils the graphing of the solution sets of the following open sentences in one variable on a number line.

$x + 4 = 9$

$x - 3 = 5$

$2x + 5 = 13$

$x = 7$

$x > 5$

$x - 2 < 16$

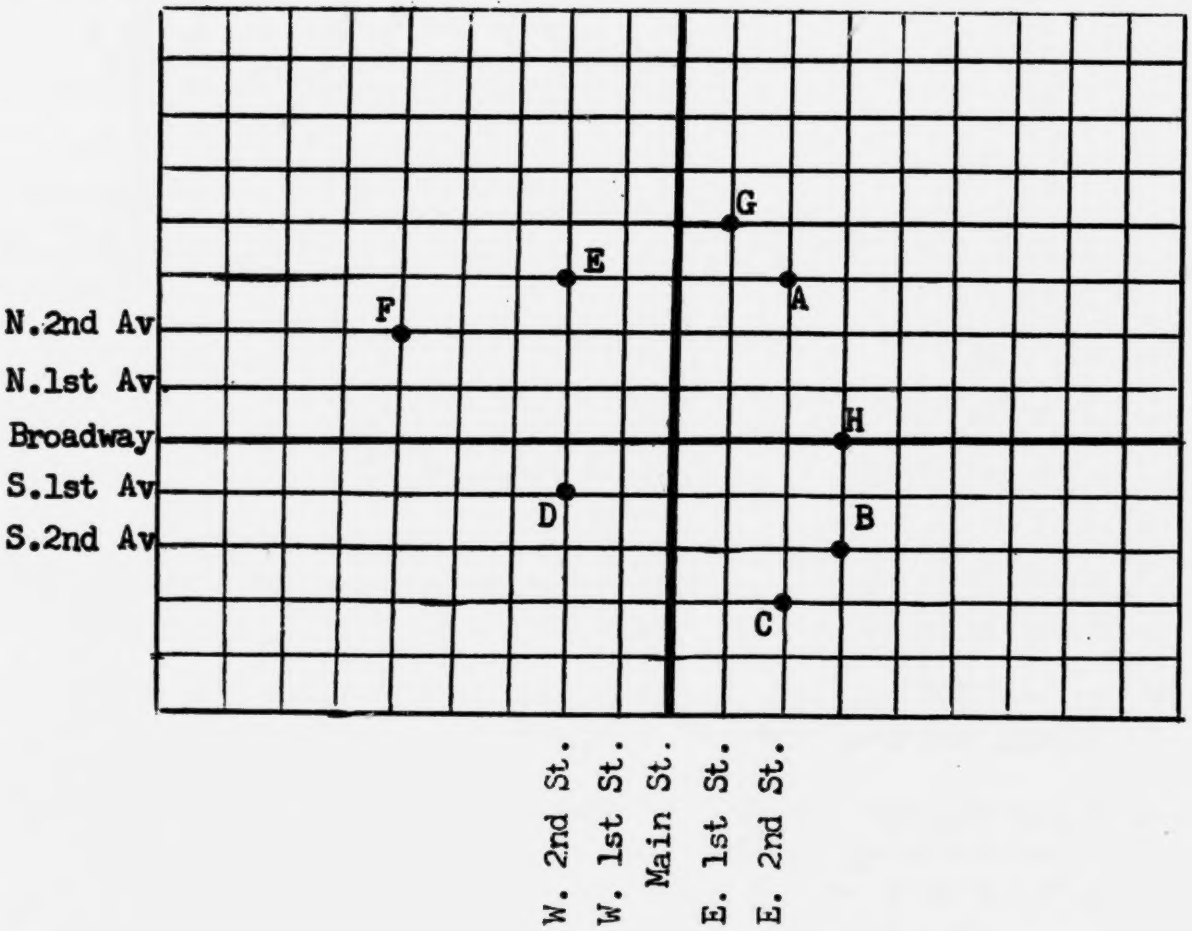
$x < 4$

$2x - x > 9$

2. Elicit that a solution of an open sentence in two variables is an ordered pair of numbers.

3. Guide pupils to an understanding of how an ordered pair of numbers may be associated with a point in a number plane as follows:

   a. Have pupils examine a map of an imaginary town such as the following:


G
E
A
F
N.2nd Av
N.1st Av
Broadway
H
S.1st Av
D
S.2nd Av
B
C
W. 2nd St.
W. 1st St.
Main St.
E. 1st St.
E. 2nd St.

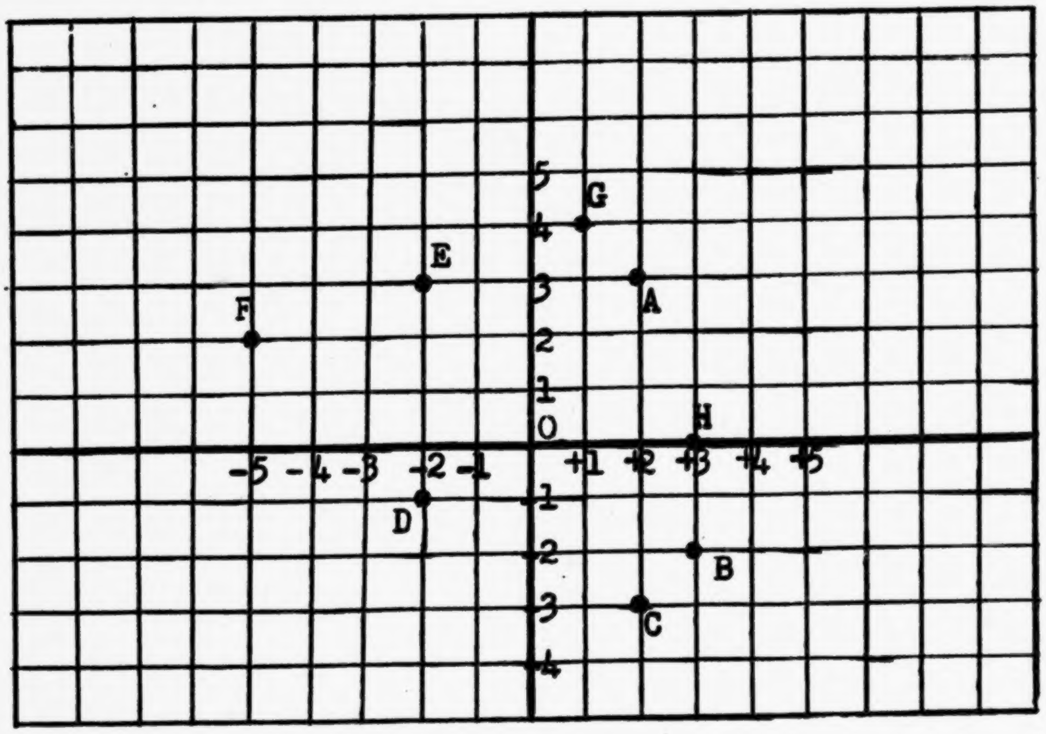
Note to teacher: If preferred, pupils may devise a map using local streets and places of interest.

Have pupils observe that the streets and avenues cross at right angles. Have them suggest the directions that must be given to find the following places in the town, using the intersection of Main Street and Broadway as a reference or starting point:

A. Art Museum (2 blocks east and then 3 blocks north)	E. Eastern S.S. Office
B. Bayview High School	F. Fairmont Hospital
C. City Hall	G. General Post Office
D. Dan's Supermarket	H. Hall of Records

Elicit the number of directives (2) that must be given to find each place. Have pupils consider what is implied when only one directive is given.

b. Guide pupils to see that Main Street and Broadway may be thought of as a pair of number lines which intersect at right angles. The point where the two number lines meet is called the origin.



Have pupils give instructions for finding the Art Museum (start at the origin, go two blocks east, then three blocks north). Elicit a method of recording these instructions very briefly using signed numbers. We write "two blocks east" as +2, "three blocks north" as +3, and enclose them in parentheses (+2,+3).

Have pupils note that this is an ordered number pair which will locate the Art Museum correctly if the first number, +2, is used as an east-west directive, and the second number, +3, is used as a north-south directive. Have pupils try locating the Art Museum when the order of the numbers is reversed. Elicit that if we agree that the first number is always to be an east-west directive, and the second number a north-south directive, then each point on the map is associated with one, and only one, number pair.

c. Have pupils state the number pair that describes the location of

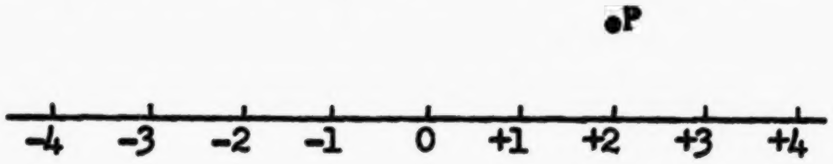
Dan's Supermarket
General Post Office
City Hall
Fairmont Hospital, etc.

d. Have pupils see that although the number pair (5,-1) carries the directives "5 blocks east, then 1 block south," it makes no difference in what order we carry out the directives. That is, (5,-1) can be located just as well by the directives "1 block south, then 5 blocks east." We must remember, however, that the first number of the ordered pair is an east-west number, and the second, a north-south number.

e. Have pupils realize that by means of two number lines drawn at right angles to each other, we can associate points in a plane with ordered number pairs.

1) Have pupils consider the following:

Suppose P is a point in a plane that is not on a number line drawn in the plane.



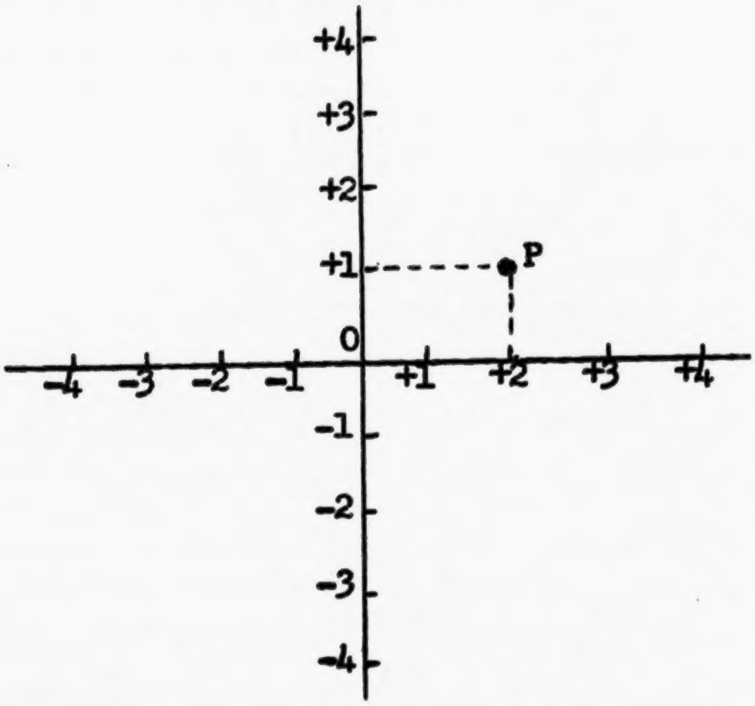
Have them note that P is directly above (or below) some point of the number line.

How can we find what point on the number line P is directly above? (Draw a line through P perpendicular to the number line.)

In the above diagram, P is directly above the point on the number line which corresponds to the number 2. Which other points in the plane are directly above this same point on the number line? (all other points on the perpendicular)

How can we distinguish the position of P from that of all these other points?

2) Have pupils see that we can solve this problem by drawing a second number line at right angles to the first in such a way that their zero points coincide.



Have them note that P is directly to the right (or left) of some point on this second number line.

How can we find what point on the second number line P is directly to the right of? (Draw a line through P perpendicular to the vertical number line.) In the foregoing diagram, P is directly to the right of the point which corresponds to the number 1 on the vertical number line.

3) Have pupils realize that point P, which is two units to the right of the vertical line, and one unit above the horizontal line, may be represented by the ordered number pair (2,1). The two numbers in a pair assigned to a point

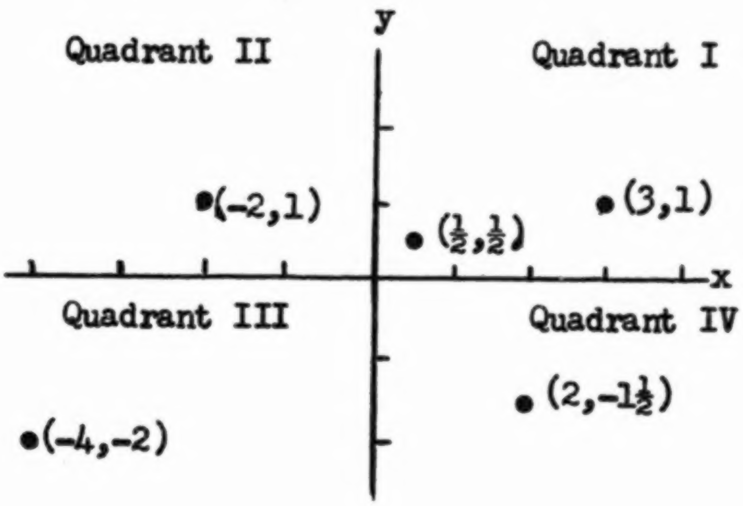
are called the coordinates of the point. The first coordinate gives the directed distance of a point from the vertical number line, and the second coordinate gives the directed distance from the horizontal number line.

f. Have pupils state the coordinates of points A, B, C, D, E, F, G, H on the map (see page 7 . Point H has a first coordinate of +3 and a second coordinate of 0). Have pupils describe the coordinates of several points on the horizontal (E-W) number line, on the vertical (N-S) number line. Elicit which of the points A, B, ... H have the same first coordinate; the same second coordinate.

g. Tell pupils that this system of assigning ordered number pairs to points in a number plane is called a cartesian coordinate system in honor of the 17th century French mathematician, Rene Descartes, who devised the system.

h. Tell pupils that the symbol "x" is used for the first coordinate and the symbol "y" for the second coordinate of a variable point in a plane. The horizontal number line is called the x-axis and the vertical number line the y-axis of the cartesian coordinate system. The point of intersection of the axes is called the origin. The first number in each ordered pair is called the x-coordinate or abscissa; the second number, the y-coordinate or ordinate of the corresponding point. The coordinates of the origin are (0,0).

i. Have pupils note that just as folding a piece of paper lengthwise down the middle, and then horizontally across the middle divides the paper into four parts, so the x-axis and y-axis divide the number plane into four parts. Each part is called a quadrant, and the quadrants are numbered counterclockwise, from I to IV, starting with the upper right quadrant, as indicated in the figure.



j. Have pupils see that a coordinate system enables us to find a particular point in a plane which corresponds to an ordered pair of numbers. The point is called the graph of the ordered pair. To locate the graph of the ordered pair (-2,1), we visualize a vertical line through the graph of(-2,0) (a point on the x-axis) and a horizontal line through the graph of (0,1) (a point on the y-axis). The point of intersection of these lines is the graph of (-2,1). Locating the point and marking it with a dot is called "plotting" the point. Several points are plotted in the figure above.

B. Suggested Practice

1. Name the quadrant containing the points whose coordinates are: (2,3), (4,-2), (-9,3), (-4,-5), ($\frac{1}{2}$,1), ($-\frac{2}{3}$,$-\frac{1}{5}$)

2. Name the quadrants in which the points described below are located:

   the abscissa is 5 (Quadrants 1 and 4) the abscissa is -1
   the ordinate is 7 (Quadrants 1 and 2) the ordinate is -8

3. Graph each of the ordered number pairs:

   a. (1,4)
   b. (2,0)
   c. (-3,3)
   d. (-6,-9)
   e. (0,0)
   f. (7,$\frac{1}{2}$)
   g. (-1$\frac{1}{2}$,3)
   h. (0,-2$\frac{1}{4}$)

4. Plot each of the points:

   A(-3,-1), B(5,-1), C(5,2) D(-3,2)

   Connect the points A, B, C, D, A in order. What is the perimeter of the resulting figure? What is its area?

5. Plot each of the points:

   A(0,0) B(4,4) C(0,8)

   Join the points A,B,C,A, in order, and find the area of the resulting figure.

6. The following are three vertices of a rectangle. Find the fourth vertex.

   (-2,0), (-2,2), (3,2)

7. Determine, by graphing, whether the points (-1,1), (1,-1), (3,-3) lie on a straight line. If so, find the coordinates of three other points on this line.

8. How would you describe the location of a point if

   a. its abscissa is negative?
   b. its ordinate is positive?
   c. its abscissa is positive and its ordinate is negative?
   d. its abscissa is negative and its ordinate is positive?

9. Describe the coordinates of a point if

   a. the point is in Quadrant I
   b. the point is in Quadrant II
   c. the point is in Quadrant III
   d. the point is in Quadrant IV
   e. the point lies on the y-axis between Quadrant I and Quadrant II
   f. the point lies on the x-axis between Quadrant II and Quadrant III

10. What is the abscissa of every point on the y-axis?
    What is the ordinate of every point on the x-axis?

III. The Graph of a Linear Equation in Two Variables

A. Suggested Procedure

1. Review the use of the number line to graph an equation in one variable. (The domain of the variable is the set of signed numbers.)

   $x = 3$ $\qquad$ $x + 1 = 5$ $\qquad$ $4x = 2x + 4$ $\qquad$ $2x + 4 = x - 2$

   a. Have pupils recall that the graph of an equation in one variable is the graph of its solution set.

   b. Elicit that the graph of each of the above equations is a unique point on the number line.

2. Pose question: Consider the equation in two variables $x + y = 3$. What would its graph look like on the coordinate plane?

   a. Review the understanding that, with the value of x given first, an ordered pair of numbers which makes the sentence true is called a solution of the open sentence. Thus, the ordered pair (1,2) is a solution of $x + y = 3$, but the ordered pair (2,5) is not.

   b. Have pupils understand that we also speak of the coordinates of a point as making a sentence true (or false) if, when the coordinates are substituted appropriately, the sentence becomes true (or false).

   c. Guide pupils to realize that the graph of the sentence $y = x + 2$, for example, is the set of all points on the plane whose coordinates make the sentence true. That is to say, the graph of an open sentence in two variables is the graph of its solution set.

3. Have pupils learn how to use the coordinate plane to graph simple equations in two variables.

   a. Review method of finding solutions of an open sentence in two variables, e.g., finding solutions of $x + y = 3$. (The replacement set for both x and y is the set of signed numbers.)

1) Change $x + y = 3$ into an equivalent equation which expresses y in terms of x: $y = 3 - x$.

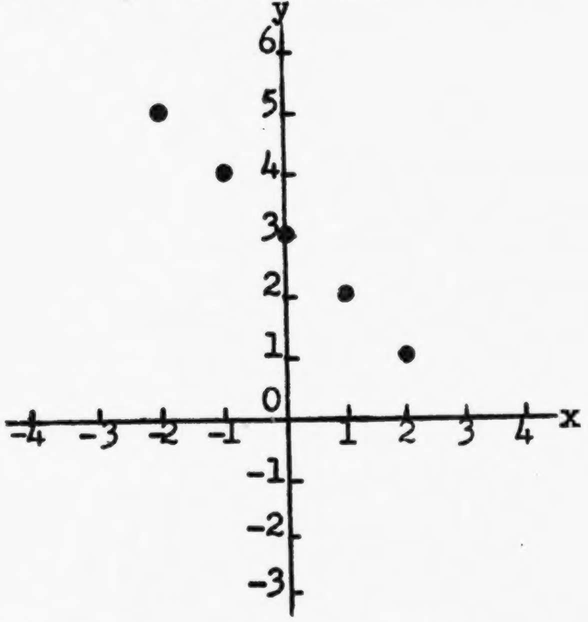
2) Make any signed number replacements for x and calculate the corresponding values of y.

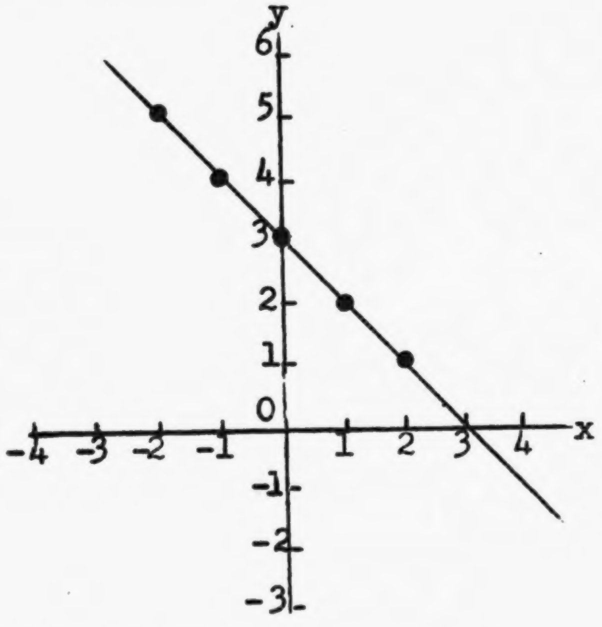
x	3 - x	y
-2	3 - (-2)	5
-1	3 - (-1)	4
0	3 - (0)	3
1	3 - (1)	2
2	3 - (2)	1

(-2,5), (-1,4), (0,3), (1,2), (2,1) are some solutions of the equation.

Although we cannot find all the solutions, we would like to know what picture their graph presents.

b. Have pupils plot the points which correspond to the solutions of $x + y = 3$ determined above.





Have pupils note that all the points plotted appear to lie on a straight line. Have them draw the line.

c. Have pupils choose several other points (some, fairly close together) that lie on the line, e.g., the points

$(\frac{1}{4}, 2\frac{3}{4})$ $(\frac{1}{2}, 2\frac{1}{2})$ $(-1\frac{1}{2}, 4\frac{1}{2})$

Have them note that the coordinates of these points are solutions of the equation $x + y = 3$. Have them test a point such as $(1\frac{1}{2},1\frac{1}{2})$, midway between two points on the graph to see whether the number pair is a solution of $x + y = 3$. They find that it is.

d. Have pupils pick several points that are not on the line, e.g., $(4,1)$, $(2\frac{1}{2},1\frac{1}{2})$. For each such point have them test the coordinates to see whether they are solutions of the equation $x + y = 3$. They are not. Elicit that this line appears to be the set of all those points and only those points whose coordinates satisfy the equation. Tell pupils that this is actually so, although we are not ready at this stage to prove it mathematically.

Note: At present, pupils' knowledge is limited to the set of rational numbers. The continuity of a line, however, implies the existence of the real number system, including irrational numbers, as well as rationals.

e. Tell pupils that the line is called the graph of the equation $x + y = 3$. The open sentence $x + y = 3$ is called an equation of the line.

Note: Each line has many equivalent equations:

$2x + 2y = 6$, $-3x -3y = -9$, $\frac{x}{2} + \frac{y}{2} = \frac{3}{2}$ are all equations of the line in the figure. It should also be emphasized that the drawings are incomplete pictures. The line should extend indefinitely in both directions.

f. Tell pupils that an equation, such as those we have been considering, of the form $ax + by = c$, where a and b are not both 0, is called a linear equation. Its graph is a straight line.

4. Have pupils organize work in graphing a linear equation in two variables.

a. One convenient procedure is as follows:

Graph the equation $x - y = -3$

1) Change $x - y = -3$ into an equivalent equation which expresses y in terms of x (or x in terms of y).

$$\begin{aligned} x - y &= -3 \\ -y &= -3 - x \\ y &= 3 + x \end{aligned}$$

2) Make several replacements for x and calculate y.

x	3 + x	y
-2	3 + (-2)	1
0	3 + 0	3
3	3 + 3	6
5	3 + 5	8

(-2,1), (0,3), (3,6), (5,8) are some solutions of the equation.

3) On graph paper, draw a pair of coordinate axes and plot the number pairs in 2). Note that they seem to lie on a straight line. Draw the line.

b. Discuss with pupils the number of ordered pairs needed to draw the graph of a linear equation. Whereas only two points are required to determine a line, using three or four number pairs may help reveal an error in computation.

Note: To avoid having pupils think that the graphs of all equations are straight lines, have them try to graph $y = x^2$ for $x = 0, 1, 2, 3$.

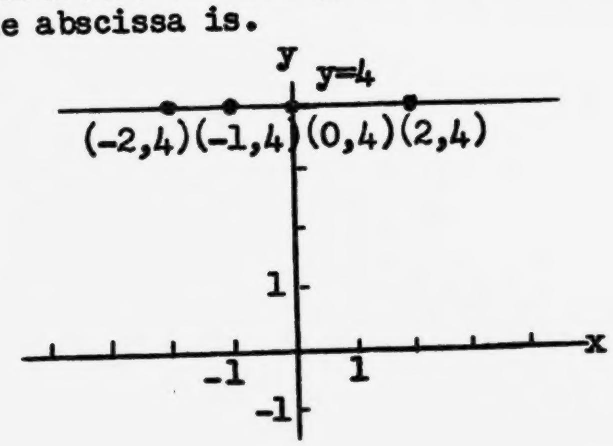
5. Guide pupils to realize that the graph of an equation of the form $y = b$ ($x = a$) is a line parallel to the x-axis (y-axis).

a. Have pupils consider the open sentence $y = 4$. This may be written as an open sentence in two variables, as follows:

$$0 \cdot x + y = 4$$

Some of the solutions of this equation are (0,4), (-1,4), (2,4), (-2,4), etc. Each member of the solution set is of the form (x,4) where the replacement for x may be any signed number.

b. Have pupils represent several solutions as points in the coordinate plane. They observe that each point has an ordinate of 4, no matter what the abscissa is.



Have pupils note that the graph of $y = 4$ is a line parallel to the x-axis. The ordinate of every point on the line is 4.

c. In a similar manner, guide pupils to realize that the graph of $x = 2$ can be obtained by considering it as an open sentence of the form $x + 0 \cdot y = 2$. Each member of the solution set is of the form $(2, y)$. The graph of $x = 2$ is a line parallel to the y-axis. The abscissa of every point on the line is 2.

B. Suggested Practice

1. In each of the following equations, solve for y in terms of x; for x in terms of y.

$x + y = 8$
$x - y = 4$
$2x + y = 10$
$x = 2y$
$y + x = 0$
$\frac{y}{4} = x$

2. Tell whether the given point is on the graph of the equation.

$y + x = 2$	$(1,1)$	Yes, because $1 + 1 = 2$
$x + 2y = 5$	$(1,4)$	
$y - 2x = 9$	$(4,1)$	No, because $1 - 8 \neq 9$
$3x + y = 1\frac{1}{4}$	$(\frac{1}{2}.\frac{1}{4})$	
$4x + 2y = 0$	$(-1,2)$	
$\frac{x}{2} - y = 5$	$(4,-3)$	
$x = 4$	$(4,5)$	
$y = 2x$	$(0,0)$	

3. Describe in words, using terms abscissa and ordinate, the relationship expressed between x and y in the following equations.

a. $y = x$ (the ordinate equals the abscissa)
b. $y = x - 2$
c. $y = \frac{x}{3}$
d. $y = 3x + 1$
e. $y - x = 2$
f. $x = \frac{y}{3}$
g. $x + y = 0$

4. Write each of the following as an open sentence:

a. the ordinate is three times the abscissa
b. the ordinate is equal to 3 more than the abscissa
c. the abscissa is one half the ordinate
d. the sum of twice the abscissa and three times the ordinate equals 5.

5. Graph each of the following equations:

a. $y = 2x$	f. $2y = x$
b. $x + y = 6$	g. $2y + 4x = 8$
c. $y = x - 3$	h. $y = -x$
d. $y = 5$	i. $y = 3x + 1$
e. $y = 0$ What is the usual name of this line?	j. $x = -2$
	k. $x = 0$ What is the usual name of this line?

6. Tell, without drawing the graph, which of the following equations have graphs which pass through the origin:

a. $y = x$ Solution: Since (0,0), the coordinates of the origin, satisfies $y = x$, then the graph of $y = x$ contains the origin.

b. $y = 4x$	e. $y = 6$
c. $x + y = 10$	f. $x = -5$
d. $3y = x$	g. $x - 3y = 8$
	h. $x + y = 0$

7. What are the coordinates of the point at which the graph of each of the following equations crosses:

a. the x-axis b. the y-axis

1) $x - y = 6$ Solution: The ordinate of every point on the x-axis is 0. Then if we assign the value 0 to y in $x - y = 6$, we obtain $x = 6$. Then $x - y = 6$ crosses the x-axis at (6,0).

2) $x - 2y = 4$
3) $3y + 4x = 12$
4) $x - y = 2\frac{1}{2}$
5) $y = 2x$

8. Tell, without drawing the graph, whether the graphs of the following equations cross the y-axis above, through, or below the origin.

a. $x + 2y = 4$ Solution: The abscissa of every point on the y-axis is 0. Then if we assign the value 0 to x in $x + 2y = 4$, we obtain $2y = 4$, or $y = 2$. The graph of $x + 2y = 4$ crosses the y-axis at (0,2) which is above the origin.

b. $y = \frac{1}{2}x$
c. $x - y = 14$
d. $y = -2$
e. $y = +7$
f. $2x + 2y = 0$

IV. Graphing an Inequality

A. Suggested Procedure

1. Review the use of the number line to graph an inequality in one variable (the domain of x is the set of signed numbers):

   $x \neq 2$ $\qquad$ $x < 3$
   $x > 5$ $\qquad$ $x + 1 > 4$

   a. Have pupils recall that the graph of an inequality is the graph of its solution set.

   b. Elicit that, depending on the domain of the variable, the graph of an inequality in one variable is represented by one or more points on the number line.

2. Pose question: Consider an inequality in two variables, such as $y > x + 1$. What would its graph look like on the number plane?

3. Have pupils learn how to use the coordinate plane to graph simple inequalities in two variables.

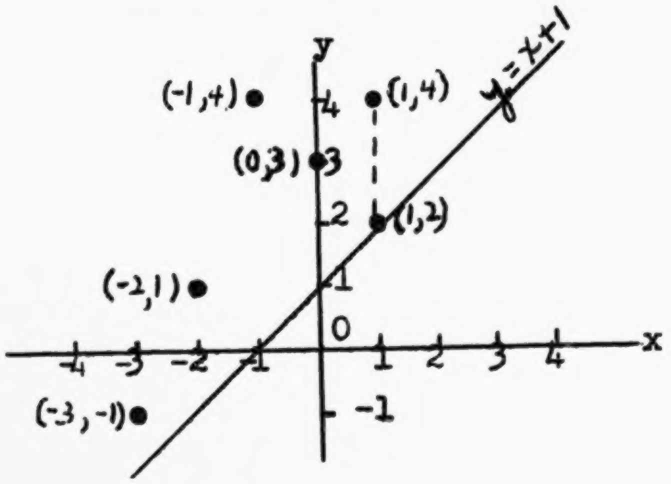
   a. Have them draw the graph of the related equality $y = x + 1$. The graph is made by finding several sample ordered pairs and plotting them on the coordinate plane. The points corresponding to the ordered pairs lie on a straight line. Elicit that all the points on the line, and only these, satisfy the equation $y = x + 1$. The line divides the number plane into two half-planes.

   b. Discuss with pupils how we can graph the inequality $y > x + 1$. Can we find all the ordered pairs that satisfy the inequality and plot them on a coordinate plane? Will sample ordered pairs satisfying the inequality help us determine the complete graph?

   c. Have pupils choose some sample ordered pairs satisfying the inequality $y > x + 1$, as, for example,

      (0,3) $\qquad$ (-1,4) $\qquad$ (-3,-1) $\qquad$ (-2,1)

      Have them plot these sample ordered pairs on the same coordinate system that was used to graph the equation $y = x + 1$.



Have pupils note that all the sample ordered pairs which satisfy the inequality $y > x + 1$ are associated with points in the region of the coordinate plane "above" the line.

Have them test other points above the line, as for example, the point corresponding to (1,4). Substituting this ordered pair in the open sentence $y > x + 1$, we obtain $4 > 1 + 1$, which is a true statement.

Have them observe that any point between (1,4) (which is above the line), and (1,2) (which is on the line), has coordinates which satisfy the inequality, e.g.,

(1,3)	$3 > 1 + 1$	true statement
$(1,2\frac{1}{2})$	$2\frac{1}{2} > 1 + 1$	true statement
$(1,2\frac{1}{4})$	$2\frac{1}{4} > 1 + 1$	true statement etc.

Have pupils test points on the line, e.g.,

(1,2)	$2 > 1 + 1$	false statement

Elicit that points on the line satisfy the equality $y = x + 1$, but do not satisfy the inequality.

Have pupils test points below the line:

$(1,1\frac{1}{2})$	$1\frac{1}{2} > 1 + 1$	false statement
(1,1)	$1 > 1 + 1$	false statement
(1,0)	$0 > 1 + 1$	false statement

Guide pupils to see that whereas the points above the line satisfy the inequality $y > x + 1$, those on the line, and those below the line, do not.

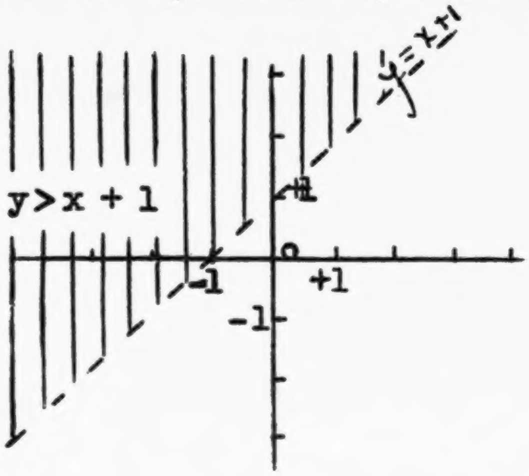
d. To have pupils conclude that the graph of $y > x + 1$ is a region composed of the set of all points that lie above the line $y = x + 1$, pose following questions:

Choose any point on the line $y = x + 1$. What is its abscissa? What is its ordinate? What is the relationship between the ordinate and the abscissa for points on the line? (The ordinate equals the abscissa increased by 1, i.e., $y = x + 1$.)

Consider the set of points which have the same abscissa as the chosen point, but whose ordinates are greater than the ordinates of that point. Where are these points located? (above the line) What is the relationship between the ordinate and the abscissa of each point in this set. (The ordinate is greater than the abscissa increased by 1, i.e., $y > x + 1$.)

Pose similar questions for the set of points which have the same abscissa as the chosen point on the line, but whose ordinates are less than the ordinate of that point.

Have pupils realize that the graphs of $y > x + 1$ is a region composed of the set of all points that lie above the line $y = x + 1$. It can be represented by shading the region as shown.



Have pupils understand that the shaded portion representing $y > x + 1$ does not include the line $y = x + 1$. The line serves as a boundary for the region and is shown as a dashed line to indicate it is not included.

Note: If the graph is that of a combined equation and inequality, e.g., $y \geq x + 1$, then the points of the line $y = x + 1$ are included with the points representing $y > x + 1$. In such a case, a solid line is used to represent $y = x + 1$ and indicates that the shaded portion actually includes the boundary line.

e. In a similar manner have pupils choose sample ordered pairs which satisfy the inequality $y < x + 1$ and plot them on the coordinate plane. Guide them to see, using procedures analogous to IV-2-a-c, that the graph of this inequality is a region composed of the sets of all points that lie below the line $y = x + 1$.

f. Review with pupils that because two points determine a line, we need only two number pairs which satisfy a linear equation to determine the graph of the equation. (Three or four pairs are generally used as a precaution against error.) How many sample ordered pairs do we need to determine the region of the graph of an inequality?

Guide pupils to see that plotting one sample ordered pair which satisfies the inequality is sufficient to determine the region of the graph of the inequality. This is so because all ordered pairs which satisfy the inequality, and only these, represent points on the same side of the line which is the graph of the related equality. However, it is advisable to use a second sample ordered pair as a check.

g. By using procedures similar to III-A-6, have pupils realize that the graph of an inequality of the form $y > b (y < b)$ is the half-plane above (below) the line $y = b$; the graph of an inequality of the form $x > a (x < a)$ is the half-plane to the right (left) of the line $x = a$.

4. Have pupils summarize the procedures in graphing an inequality in two variables, as follows:

   a. Replace the inequality symbol ($<$ or $>$) with an equality symbol and then graph the resulting equation. Represent the line as a series of dashes.

   b. Choose two sample ordered pairs (one as a check) which satisfy the inequality. Plot these ordered pairs as points in the coordinate plane. Note on which side of the line the two points lie. The graph of the inequality is the half-plane lying on that side of the graphed line. Shade this region.

   c. In graphing a combined equation and inequality ($\leq$ or $\geq$), the graph is a half-plane and the boundary line (drawn as a solid line).

B. Suggested Practice

Graph the following inequalities:

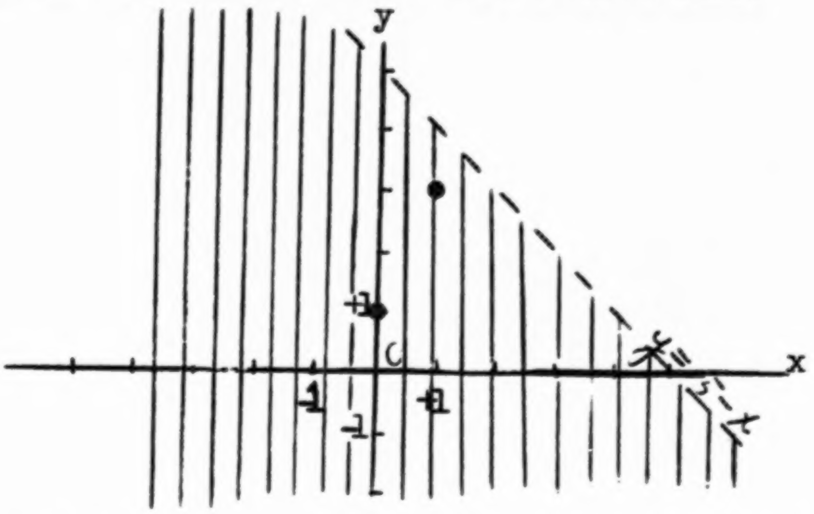
1. $x + y < 5$

   Solution

   a. Transform the inequality into an equivalent inequality having y as one member.

      $y < 5 - x$

b. Graph $y = 5 - x$ and show it as a dashed line.



c. Choose two sample ordered pairs, e.g., $(1,3)$ and $(0,1)$ which satisfy the inequality. Plot these as points on the coordinate plane. Note on which side of the line they lie.

d. Shade the half-plane on this side of the line. The shaded region is the graph of the inequality $x + y < 5$.

2. $y > 2x - 3$
3. $x > 3y$
4. $y < 5x$
5. $x < y + 8$
6. $2x + y > 3$
7. $x \geq 3y$
8. $y \geq 2x - 3$
9. $y \leq 5x$

V. Meaning of Systems of Equations

A. Suggested Procedure

1. Meaning of systems of linear equations

Have pupils understand the meaning of systems of linear equations.

a. Pose problem: The sum of two numbers is 12. What are the numbers?

Have pupils describe this condition as an open sentence in two variables.

Let x = one number
y = other number
Then
$x + y = 12$

b. Elicit that there are infinitely many pairs of numbers in the solution set of the sentence $x + y = 12$. For a unique answer, a second condition must be imposed upon the numbers.

c. Add a second condition (for example, the difference of the two numbers is 4). The problem now reads: the sum of two numbers is 12; their difference is 4. What are the numbers?

Have pupils describe each condition of the problem as an open sentence in two variables:

$$x + y = 12$$
$$x - y = 4$$

d. Elicit that there are infinitely many number pairs in the solution sets of each of the above sentences.

Some of the number pairs in the solution set of $x + y = 12$ are: (1,11), (4,8), (8,4), (6,6), (-2,14), (19,-7).

Some of the number pairs in the solution set of $x - y = 4$ are: (9,5), (10,6), (8,4), (-12,-16), (-2,-6).

e. Have pupils note, by considering these ordered pairs, that (8,4) is in both sets. We have "solved" the problem of finding a pair of numbers that satisfies both conditions.

Note: It is not apparent from studying the solution sets above whether this is the only pair common to both sets. However, in this particular example it is.

f. Tell pupils that when the conditions of a problem are expressed by two linear equations, and these conditions must hold simultaneously, the set of equations is called a system of linear equations. To solve such a system, i.e., to find its solution set, we want to find the set of ordered number pairs, each of which is a solution of both equations.

2. Have pupils realize that for certain systems of linear equations it is possible to find the solution set with very little work.

a. Find the solution set of the following system:

$$x = 2$$
$$y = 5$$

The only replacements that make both equations true are 2 for x and 5 for y. Then, $\{(2,5)\}$ is the solution set of the system. This is a one-element set.

b. Solve the following system of linear equations:

$$\begin{aligned} x + y &= 3 \\ y &= 1 \end{aligned}$$

The only replacement for y that makes the second equation true is 1. Then the only replacement for y that can make both equations true is 1. Replacing y by 1 in the first equation, we have $x + 1 = 3$, which is an equation in one variable.

Solving, $x = 2$. Then if y is to be replaced by 1 in the first equation, x must be replaced by 2 to make the equation true. The solution set of the system is $\{(2,1)\}$.

B. Suggested Practice

Find the solution set of each of the following systems:

1. $x = 3$, $y = 1$
2. $x = 8$, $y = 4$
3. $x = 4$, $y = 0$
4. $x + y = 9$, $x = 3$
5. $4x - y = 7$, $y = 1$
6. $x = \frac{1}{2}$, $y = \frac{2}{3}$

VI. Graphical Method of Solving a System of Equations

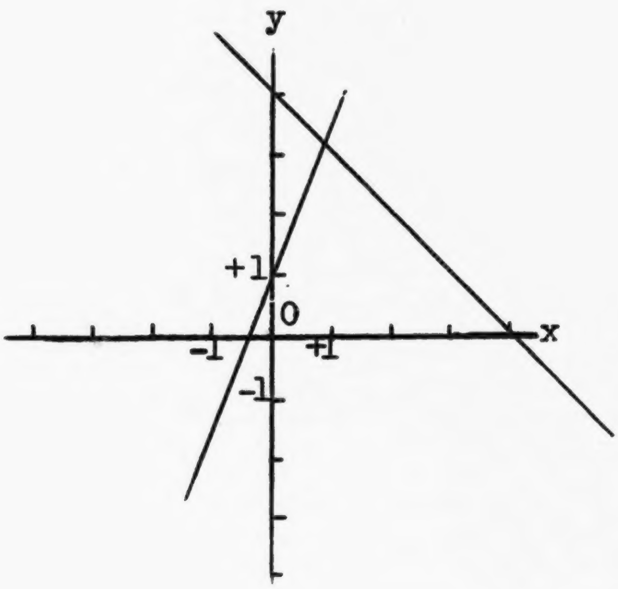
A. Suggested Procedure

1. Lead pupils to see that in attempting to solve a system of linear equations by selecting sample ordered number pairs for two tables, we might not choose the ordered pair which is the common solution. For example, if the common solution were $(\frac{1}{2},5)$, we would not be likely to use this as one of the sample ordered number pairs. Have pupils see the need for solving systems by another method, e.g., graphing.

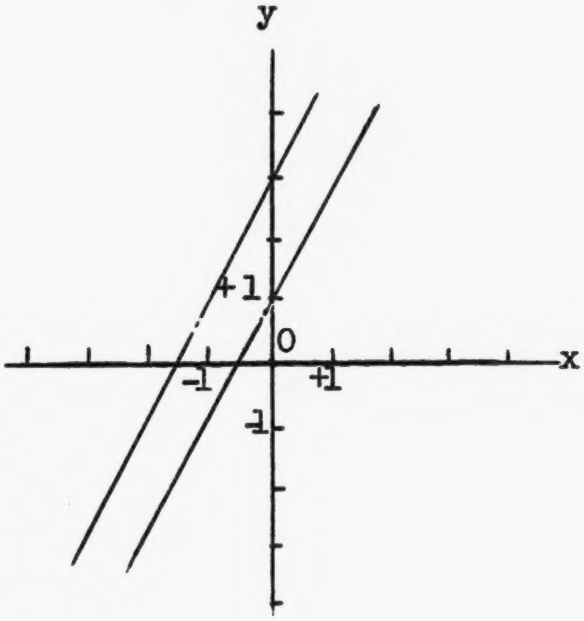
2. Discuss with pupils the fact that, if we graph each equation of a system of two linear equations using the same coordinate plane, we get two straight lines. What are the possible relationships between the two lines? Elicit that:

   a. the two lines may intersect in one point, or
   b. they may be parallel and have no point in common, or
   c. the lines may coincide, that is, they have all points in common (a single line).

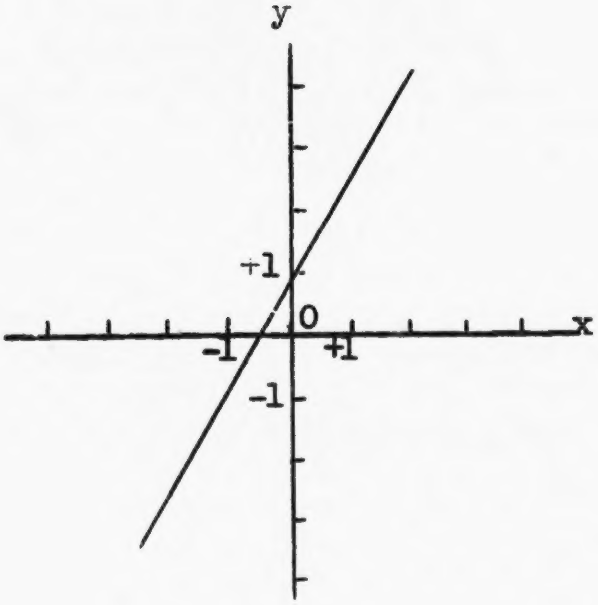
3. Have pupils examine examples of these three situations.



Intersecting Lines
$y = 2x + 1$
$y = -x + 4$



Parallel Lines
$y = 2x + 1$
$y = 2x + 3$



Coinciding Lines (Single Line)
$y = 2x + 1$
$2y = 4x + 2$

4. Have pupils study each of the three possibilities:

   a. Intersecting lines

      1) Have pupils recall that every linear equation in two variables determines an unlimited set of ordered number pairs. Have pupils find some samples of ordered number pairs which satisfy each of the equations of the two intersecting lines. These ordered number pairs are the coordinates of the points on the graph of each equation.

         $y = 2x + 1$:

x	-2	-1	0	1	2	3	4
y	-3	-1	1	3	5	7	9

         Thus, some solutions of $y = 2x + 1$ are: (-2,-3), (-1,-1), (0,1), (1,3), (2,5), (3,7), (4,9).

         $y = -x + 4$:

x	-2	-1	0	1	2	3	4
y	6	5	4	3	2	1	0

         Thus, some solutions of $y = x + 4$ are: (-2,6), (-1,5), (0,4), (1,3), (2,2), (3,1), (4,0).

         Have pupils note that there is one ordered pair, (1,3), which is the same in both sets.

      2) Discuss the number of pairs that would be alike in both sets if all ordered pairs could be examined, instead of just a few. Have pupils realize that the solution set of the system is $\{(1,3)\}$, a single-element set.

      3) Guide pupils to see that the ordered pair (1,3) corresponds to the point of intersection of the two lines. Tell pupils that equations such as these, which have at least one common solution, are called consistent equations. We also label such equations independent because their solution sets are not identical. Their graphs are different straight lines.

   b. Parallel lines

      1) Have pupils find some samples of ordered number pairs which satisfy each of the equations of the two parallel lines.

         As already determined, some solutions of $y = 2x + 1$ are: (-2,-3), (-1,-1), (0,1), (1,3), (2,5), (3,7), (4,9).

$y = 2x + 3$:

x	-2	-1	0	1	2	3	4
y	-1	1	3	5	7	9	11

Thus, some solutions of $y = 2x + 3$ are: (-2,-1), (-1,1), (0,3), (1,5), (2,7), (3,9), (4,11).

Have pupils note that there is no pair that is the same in both sets. Discuss with pupils whether the infinite set of ordered pairs for each equation would ever have a pair in common. Have pupils conclude that because parallel lines never meet, there are no ordered pairs in common. If such an ordered pair existed, its graph would be a point on both lines, but this is impossible since the lines do not intersect and thus have no common point. The solution set of this system is an empty set.

2) Tell pupils that equations of a system which have no common solution are called inconsistent equations. The graphs of these equations are parallel lines.

3) Have pupils see that, without graphing, it is possible to tell whether the equations of a system are inconsistent, i.e., have no common solution. For example, in the equations of the system

$$\begin{cases} y = 2x + 1 \\ y = 2x + 3 \end{cases}$$

the corresponding terms are alike, except for the constant term. It is not possible for a number (y) to be 1 more than twice another number (x) and also to be 3 more than twice that same number. These equations are inconsistent.

c. Coinciding lines (Single line)

1) Have pupils find sample ordered number pairs which satisfy each of the equations of the two coinciding lines.

As already determined, some solutions of $y = 2x + 1$ are: (-2,-3), (-1,-1), (0,1), (1,3), (2,5), (3,7), (4,9).

$2y = 4x + 2$:

x	-2	-1	0	1	2	3	4
y	-3	-1	1	3	5	7	9

Thus, some solutions of $2y = 4x + 2$ are: (-2,-3), (-1,-1), (0,1), (1,3), (2,5), (3,7), (4,9).

Have pupils note that the sets of ordered pairs for both equations are identical. Have them realize that when lines coincide, the set of ordered pairs which are the coordinates of points on the graph of one line must be exactly the same set as that for the other line.

2) Tell pupils that equations of a system whose solution sets are identical are called dependent equations. The graph of a dependent system of two linear equations consists of two lines that coincide (a single line).

3) Have pupils see that without drawing a graph, it is possible to tell when the equations of a system are dependent. For example, consider the system

$$\begin{aligned} y &= 2x + 1 \\ 2y &= 4x + 2 \end{aligned}$$

Ask pupils how the two equations are related. The two equations are equivalent since the second may be obtained from the first by multiplying both members of the first equation by 2. Equivalent equations have the same solution set and are therefore dependent. Thus, the equations of the above system are dependent.

d. Have pupils realize that a system of linear equations can be solved (if a unique solution exists), by graphing the equations of the system on the same coordinate plane, and determining the coordinates of the point of intersection.

5. Have pupils solve systems of linear equations by graphing, as follows:

Solve graphically the system of equations:

$$\begin{aligned} 2x - y &= 5 \\ x + 3y &= 6 \end{aligned}$$

a. Find several ordered number pairs (solutions) for each equation.

$2x - y = 5$

x	0	2	4	-2
y	-5	-1	3	-9

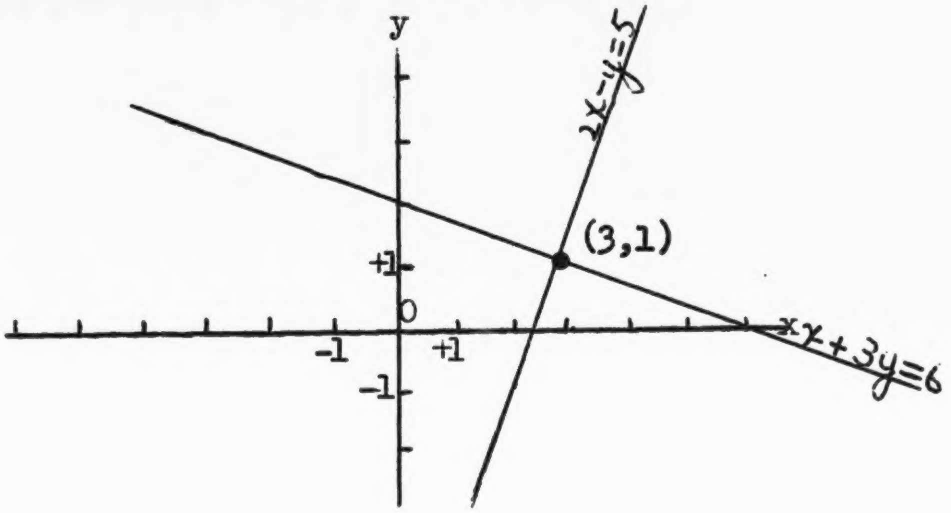
(0,-5), (2,-1), (4,3), (-2,-9) are some solutions of $2x-y=5$

$x + 3y = 6$

x	0	3	6	-3
y	2	1	0	3

(0,2), (3,1), (6,0), (-3,3) are some solutions of $x+3y=6$

b. Graph each equation on the same coordinate plane.



c. From the graph it appears that the ordered pair (3,1) is the common solution.

d. Check the apparent solution (3,1) by verifying that the ordered pair (3,1) satisfies both equations.

$$\begin{aligned} 2x - y &= 5 \\ 2(3) - 1 &\overset{?}{=} 5 \\ 5 &= 5 \end{aligned} \qquad\qquad \begin{aligned} x + 3y &= 6 \\ 3 + 3 &\overset{?}{=} 6 \\ 6 &= 6 \end{aligned}$$

We conclude the solution of the system is (3,1). The solution set is $\{(3,1)\}$.

B. Suggested Practice

1. Identify the equations in these systems as consistent or inconsistent, dependent or independent. If possible, decide without drawing a graph.

   a. $x + y = 5$
      $2x + 3y = 12$

   b. $2x - y = 7$
      $2x - y = 4$

   c. $6x = 4y - 2$
      $3x - 2y = -1$

   d. $3x + 3y = 4$
      $2y = 3x - 4$

   e. $x + y = 1$
      $2x + 2y = 3$

   f. $\frac{1}{2}x + y - 3 = 0$
      $2x + 4y = 12$

2. Solve the following systems of linear equations graphically. Check your answers.

   a. $x - y = 3$
      $x + y = 5$

   b. $x - 2y = 4$
      $3x - 5y = 8$

   c. $x - 5 = 0$
      $y + 2 = 0$

   d. $y = 2x - 8$
      $x - 2y = 4$

   e. $y = 2x - 6$
      $3x + 3y = 9$

   f. $x - y + 1 = 0$
      $x + y = 3$

   g. $\frac{x}{2} - \frac{y}{5} = 2$
      $2x + y = -1$

   h. $2x - y + 1 = 0$
      $3y - 4x = 4$

3. Write a system of equations in two variables for each of the following problems, and then solve graphically. Check your answers.

   a. The sum of two integers is 14. When the larger integer is subtracted from twice the smaller, the result is 4. What are the integers?

   b. Two times a number plus a second number is 13. Two times the second number plus the first is 17. What are the numbers?

c. A truck left a town and traveled eastward at an average speed of 30 mph. Two hours later, a second truck left the town traveling 40 mph. in the same direction. In how many hours will the second truck overtake the first? How far from the town will they be?

Illustrative solution:

Let x = number of hours the first truck traveled
y = number of miles traveled by each truck

The conditions of the problem can be expressed by the following system of equations:

$$\begin{aligned} 30x &= y \\ 40(x-2) &= y \end{aligned}$$

The solution set of the system can be found by graphing.

Note: The points on the x-axis represent the number of hours of travel, and the points on the y-axis represent the number of miles traveled. For convenience, a unit on the y-axis may represent a number of miles.

d. A boy starts along a bicycle path riding at the rate of 3 miles an hour. One hour later his friend starts along the same path riding at the rate of five miles an hour. In how many hours will the second boy overtake the first? How far from the starting point will they be?

e. A triangle has sides that lie on the graphs of the equations $2y + 3x = 16$, $y = 2$, $y + 3x = 2$. Graph the equations and find the ordered pairs that represent the vertices of the triangle.

f. In the following system of equations, what relationship between a and b will make the system inconsistent?

$$\begin{aligned} ax + y &= 6 \\ bx + y &= 2 \end{aligned}$$

g. What values of a and b will make this system dependent?

$$\begin{aligned} 4x &= 2y + 8 \\ ax &= y + b \end{aligned}$$

h. A point whose ordinate is twice its abscissa lies on the graph of $3x + y = 10$. Find the coordinates of the point.

VII. Systems of Equations Solved by Substitution

A. Suggested Procedure

1. Review with pupils that a system of two linear equations has a solution when the lines representing the graphs of the equations

intersect. Discuss the difficulty of using the graphical method for solving a system of equations when the numbers in the solution are not integers, e.g.,

$$(4\tfrac{2}{3}, 3\tfrac{1}{4})$$

It is difficult to read the values of the numbers in the above solution from the graph with accuracy. It is, therefore, necessary to find a method for solving a system of equations that will avoid the possible inaccuracies of the graphical method.

2. Have pupils learn to solve systems of equations by the substitution method.

   a. Have them consider the equations:

      1) $y = 2x + 1$

      2) $3x + 2y = 9$

      From previous work the pupils know that these equations are not inconsistent or dependent and, therefore, there is a unique ordered number pair which is the solution of this system of equations.

      When the first and second members of the solution are substituted for x and y respectively in equations 1) and 2), then both 1) and 2) become true statements. In equation 1), the polynomials y and $2x + 1$ will become names for the same number. In equation 2) the polynomial $3x + 2y$ will become a name for the number 9.

   b. Have pupils see that equation 2) can be expressed as an equation in one variable by replacing y by $2x + 1$. The result is the equation $3x + 2\,(2x+1) = 9$, from which we obtain the equations

      $$\begin{aligned} 3x + 4x + 2 &= 9 \\ 7x &= 7 \\ x &= 1 \end{aligned}$$

      Therefore, 1 is the first number of the unique ordered pair which is the solution of the original system. Replace x by 1 in either of the original equations.

      $$\begin{aligned} y &= 2x + 1 \\ y &= 2 + 1 \\ y &= 3 \end{aligned} \qquad \text{or} \qquad \begin{aligned} 3x + 2y &= 9 \\ 3 + 2y &= 9 \\ 2y &= 6 \\ y &= 3 \end{aligned}$$

Therefore, the solution set of the system is $\{(1,3)\}$. The solution should be checked in both of the original equations.

c. Have the pupils use this method of substitution to solve pairs of equations, such as the following:

$$\begin{aligned} y &= 3x - 7 \\ 2x + 3y &= 12 \end{aligned}$$

$$\begin{aligned} x &= 3y - 1 \\ 2x + 3y &= 8 \end{aligned}$$

d. Now have the pupils consider the pair of equations:

$$\begin{aligned} 3x + y &= 19 \\ 4x - 2y &= 12 \end{aligned}$$

Guide them to realize that before using the substitution method, it will be convenient to change one of the equations to an equivalent equation with x (or y) as one member. In this example $3x + y = 19$ may be transformed to $y = 19 - 3x$. The substitution method may then be applied as before.

e. Have pupils summarize the process of solving systems of linear equations in two variables by substitution, as follows:

1) Find an equation which is equivalent to one of the given equations and has y (or x) as one member.

2) Substitute this expression for y (or x) in the other equation.

3) The resulting equation, when simplified, will have as a root one number of the unique ordered pair that satisfies both of the original equations.

4) Use this number as a replacement for x (or y) in either of the original equations to find the second number of the ordered pair which is the solution of the system.

5) The solution should be checked in both of the original equations.

3. Have pupils use two equations in two variables to solve verbal problems.

a. Pose problem: The sum of two numbers is 15. The larger is 5 more than four times the smaller. Find the numbers.

Illustrative Solution

Let x = smaller number
y = larger number

1) $x + y = 15$ (The sum is 15)
2) $y = 4x + 5$ (The larger is 5 more than four times the smaller)

$x + (4x+5) = 15$ (Replace y in equation 1) by 4x+5 and solve for x)

$5x+5 = 15$
$x = 2$
$2 + y = 15$ (Substitute 2 for x in equation 1) and solve for y)
$y = 13$

The numbers are 2 and 13.

Check

The sum of 2 and 13 is 15. Thirteen is 5 more than four times 2.

b. Pose problem: Three apples and one pear cost 26¢. At the same price, two apples and five pears cost 52¢. What is the cost of each?

Illustrative Solution

Let x = cost of one apple (in cents)
y = cost of one pear (in cents)

1) $3x + y = 26$ (Three apples and one pear cost 26¢)

2) $2x + 5y = 52$ (Two apples and five pears cost 52¢)

$y = 26 - 3x$ (Solve equation 1) for y in terms of x)

$2x + 5(26-3x) = 52$ (Replace y in equation 2) by 26 - 3x)

$2x + 130 - 15x = 52$
$x = 6$

$18 + y = 26$ (Substitute 6 for x in equation 1))

$y = 8$

The cost of one apple is 6 cents. The cost of one pear is 8 cents.

Check to see if these results satisfy the conditions of the problem.

B. Suggested Practice

1. Solve each of the following systems of equations by the substitution method:

a. $y = 3x$
$3x + 2y = 18$

b. $3y + 3x = 9$
$y = -2$

c. $a = 6b$
$3a + 27 = -4b$

d. $y = 4 - x$
$7x + 2y = -2$

e. $x = -3y + 4$
$2x - y = -6$

f. $3c - d = 11$
$5d - 7c = 1$

2. Write a system of equations in two variables for each of the following problems, and then solve by the substitution method.

   a. The difference of two numbers is 4. The sum of 1 and the larger number is twice the smaller. Find the numbers.

   b. One number is 40 more than three times another. Their sum is 160. What are the numbers?

   c. A man invested \$3700, some at 4% and the rest at 3% per year. The total annual income from the investment is \$123. How much was invested at each rate?

   d. The perimeter of a rectangle is 146 feet. If the length exceeds 5 times the width by 1 foot, what are the dimensions?

   e. A family mailed a total of 50 postcards and letters during one month, at a cost of \$2.32 for postage. If the cost of mailing a letter is 5¢, and that of a postcard 4¢, how many of each were mailed?

   f. The sum of the digits of a two-digit numeral is 11. If the tens digit is 3 more than the units digit, what is the number?

VIII. Systems of Equations Solved by Addition

A. Suggested Procedure

1. Have pupils consider the equations in the following system:

$$\begin{aligned} 3x + 2y &= 10 \\ 2x - 3y &= -2 \end{aligned}$$

Have them see the method of solving systems of equations by substitution is not always a convenient one, especially if, as in the above system, the transformation for one of the variables in terms of the other results in a fractional expression. A second method, called the addition method, may be more convenient in such cases.

2. Have the pupils form an equation by adding the polynomials on the left side of the equations and by adding those on the right side:

$$(3x + 2y) + (2x - 3y) = 10 - 2$$

or $\qquad 5x - y = 8$

3. Have pupils draw the graphs of the equations in the system. Then, on the same set of axes, have them draw the graph of $5x - y = 8$. Does the graph of $5x - y = 8$ have any points in common with the graph of the equations in the system?

Have pupils see from the graph that the solution of the system also satisfies $5x - y = 8$. Is the equation $5x - y = 8$ equivalent to either of the equations in the system? (It is not, for the graph of $5x - y = 8$ does not coincide with either of the graphs of $3x + 2y = 10$ and $2x - 3y = -2$.)

4. Have pupils repeat the procedure of 2 - 3 for one or two other systems. Lead pupils to the realization that starting with a system of two consistent equations, we may form a new equation by "addition." The new equation is not equivalent to either of the original equations, but the solution of the system also satisfies the new equation. All the other solutions of the new equation are different from any of the solutions of the original equations.

5. Have pupils consider the following systems of equations and their solution sets. For each system, have them form a new equation by addition. Have pupils check to see whether the pair of numbers in the solution set of the system satisfies the new equation.

System	Solution Set	New Equation
a. $3x + 4y = 27$ $x - y = 2$	$\{(5,3)\}$	$4x + 3y = 29$
b, $2x + y = 6$ $x + y = 5$	$\{(1,4)\}$	$3x + 2y = 11$
c. $x + 2y = 4$ $2x - 2y = 2$	$\{(2,1)\}$	$3x + 0y = 6$ or $3x=6$

6. Have pupils consider how we could find the solution set of the system in 5a, if it were not known. They may suggest the following:

   a. Try various replacements for x and y in the "new equation" (obtained by "adding" the equations of the system). Find several solutions (ordered number pairs) for this equation.

   b. Test these solutions to see which, if any, satisfy the system.

   Why is this method unsatisfactory when the "new equation" has two variables?

7. Have pupils consider whether the above procedure will work any better for the system in 5-c.

a. Pose questions:

What advantage has the "new equation" in 5-c over those in 5-a and 5-b?

What is the only replacement for x in this equation that makes it a true statement? (2)

How is this replacement for x related to the solution set of the original system? (It is the first number of the ordered pair which constitutes the solution of the system.)

Now that the first number of this ordered pair is known, how can we find the second? (Use 2 as a replacement for x in either equation of the system and find the corresponding value of y.)

$$\begin{array}{rcr} 2 + 2y = 4 & \quad\text{or}\quad & 4 - 2y = 2 \\ y = 1 & & y = 1 \end{array}$$

b. Have pupils see that the solution of the system in 5-c is (2,1). The solution set is $\{(2,1)\}$. Have the pupils check the solution in both of the original equations.

c. Have pupils realize that $x = 2$ and $y = 1$ are the equations of lines passing through the point of intersection of the graphs of the two original equations.

d. Have pupils conclude that when the equation formed by adding the equations of the system has just one variable, the solution set of the system can be readily determined. Elicit that the variable y is "eliminated" because the y-terms are additive inverses. A similar procedure could have been used if the x-terms were additive inverses.

8. Have pupils practice solving each of the following systems by addition:

a. $5x - 3y = 19$, $2x + 3y = -5$ b. $2x + 5y = 20$, $-2x + 3y = 4$ c. $.3m + .2n = 5$, $.2m - .2n = 10$

$$\begin{array}{lll} \text{a. } \begin{array}{r} 5x - 3y = 19 \\ 2x + 3y = -5 \end{array} & \text{b. } \begin{array}{r} 2x + 5y = 20 \\ -2x + 3y = 4 \end{array} & \text{c. } \begin{array}{r} .3m + .2n = 5 \\ .2m - .2n = 10 \end{array} \end{array}$$

9. After several examples, have the pupils consider the system:

$$\begin{array}{r} x + 3y = 6 \\ 2x + 4y = 10 \end{array}$$

10. Ask the pupils why adding the equations does not seem to help in finding a solution. Have them see that if we added the left members and the right members, we would get the equation $3x + 7y = 16$ in which neither variable is eliminated. This equation also contains in its solution set the solution of the original pair of equations,

but it is not helpful in finding the actual solution. They should realize that since the aim of this method is to "eliminate" one of the variables, they might proceed as follows:

Multiply the left and right members of the equation $x + 3y = 6$ by -2, obtaining the equivalent equation $-2x - 6y = -12$. Then, "eliminate" one of the variables by addition.

$$\begin{aligned} -2x - 6y &= -12 \\ 2x + 4y &= 10 \end{aligned}$$

Thus, $-2x + 2x - 6y + 4y = -12 + 10$

$$\begin{aligned} -2y &= -2 \\ y &= 1 \end{aligned}$$

Replacing y by 1 in either of the original equations, we get $x = 3$. Therefore, (3,1) is the unique ordered pair which forms the solution of our equations. Again, have the pupils check the solution in both of the original equations.

11. Have pupils see that it may be necessary to choose a multiplier for each equation in the system to produce a pair of equations equivalent to the original set, which then enable us to obtain a coefficient of 0 for one of the variables when we add. For example, in solving the system

(1) $2x - 3y = 7$

(2) $3x + 4y = 10$

We may choose -3 as a multiplier for equation (1) and 2 as a multiplier for equation (2). The following equivalent equations are obtained:

$$\begin{aligned} -6x + 9y &= -21 \\ 6x + 8y &= 20 \end{aligned}$$

It is now possible to solve by addition.

B. Suggested Practice

Solve each of the following systems of equations by the addition method:

1. $x + 4y = 19$, $3x + 5y = 33$
2. $4a + 3b = -12$, $5a + 4b = -15$
3. $.7x - .3y = 1.22$, $.3x - .6y = 3.12$

Some systems of equations are easy to solve by substitution. Sometimes the method of addition is more convenient. Decide for yourself which method to use, then solve the following systems of equations:

4. $x + 2y = 5$
$2x + y = 1$

5. $.2x - y = 2$
$3x + y = 3$

6. $.4x - .3y = .6$
$.2x + .5y = 1.6$

7. $7x - 6y = 9$
$9x - 8y = 7$

8. $\frac{1}{2}x + \frac{1}{2}y = 10$
$x - y = 8$

Write a system of equations in two variables for each of the following problems, and then solve by the most convenient method:

9. Tickets to a school play cost \$1.10 for each adult and 55¢ for each child. If 330 tickets were sold for a total of \$294.25, how many tickets of each kind were sold?

10. A coin bank contains \$3.05 in dimes and nickels. If the number of dimes is 2 less than twice the number of nickels, how many coins of each type are in the bank?

11. Tom is 6 years older than Bill. Four years ago, Tom's age was 1 year less than twice Bill's age. How old is each?

12. The difference between two numbers is 8. Twice one number is 16 more than twice the other. What are the numbers?

Note: Pupils should realize that since the equations which express the conditions of the problem are dependent, we cannot obtain a unique solution.

13. Select additional problems from various textbooks.

IX. Solving Systems of Inequalities by Graphing

A. Suggested Procedure

1. Have pupils develop an understanding of the meaning of systems of inequalities.

a. Have them consider the following pair of inequalities:

$$x + y > 0$$
$$x + y < 10$$

The replacement set for both x and y in each inequality is $\{$signed numbers$\}$.

Have pupils find some sample ordered number pairs which make the first inequality true, e.g., (1,1), (2,3), (4,7), (6,9), (-2,3), (5,8), $(4\frac{1}{2},3)$.

Have them find some sample ordered number pairs which make the second inequality true, e.g., (1,1), (2,7), (-1,-2), (5,3), (3,-2), (2,3), (-2,3), $(4\frac{1}{2},3)$.

b. Guide pupils to see that there may be many ordered pairs which satisfy both inequalities, e.g., (1,1), (-2,3), $(4\frac{1}{2},3)$.

c. Have pupils understand that when two linear inequalities impose two conditions on the variables at the same time, they are called a system of linear inequalities.

2. Pose question: How can we find the solution set of a system of linear inequalities, such as

$$\begin{aligned} y &> 2x - 1 \\ x + y &< 5 \end{aligned}$$

Have pupils realize that just as graphs are used to solve systems of equations, so also can they be used in determining the solution set of systems of inequalities.

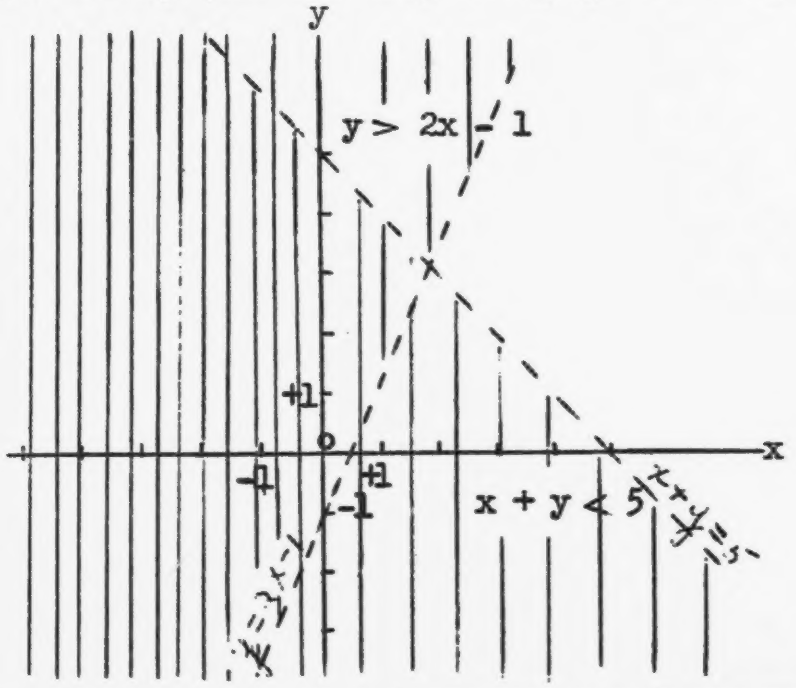
Note: Sometimes this is the only practical method of solving a system of inequalities.

3. Develop with pupils the following procedure for solving systems of inequalities by graphing:

Solve the system: $y > 2x - 1$
$x + y < 5$

a. First, draw the graph of $y > 2x - 1$. The solution set of this inequality is the set of all points in the region above, (but not on), the graph of $y = 2x - 1$.

b. Next, on the same coordinate plane, draw the graph of $x + y < 5$. The solution set of this inequality is the set of all points below, (but not on), the graph of $x + y = 5$.



Note: Since the lines $y = 2x - 1$ and $x + y = 5$ are not included, they are shown as dashed lines.

c. The graph of the solution set of the system

$$\begin{aligned} y &> 2x - 1 \\ x + y &< 5 \end{aligned}$$

is the set of all points, and only those points, on the plane which are both above the graph $y = 2x - 1$ and below the graph of $x + y = 5$. (They are indicated in the diagram by means of double-shading.) Some points in the common solution set are $(1,2)$, $(0,4)$, $(-2,-3)$. Do these points have coordinates that satisfy both sentences in the system?

OPTIONAL

4. Have pupils extend their knowledge of graphing systems of inequalities to the solution of problems involving "linear programming."

B. Suggested Practice

Graph each of the following systems of inequalities. Show the graph of the solution set as points in a double-shaded region.

1. $x > 2$
   $y < 3x$

2. $x - 1 > y$
   $y > 2 + x$

3. $y < 3x + 2$
   $y > x - 1$

4. $2y - x > 1$
   $x - 3y < 5$

5. $y - 4 \leq 0$
   $y + 2 \geq 0$

6. Graph the solution set of the following problem. Then, from the graph, find three solutions to the problem.

   I am thinking of two numbers. The first, when added to 7, results in a number less than 12. The sum of the two numbers is greater than 5. What are the numbers?

CHAPTER VIII

DIVISION OF POLYNOMIALS

This chapter contains suggested procedures for helping pupils develop understanding and skill in dividing with polynomials. The materials presented show the following progression: dividing a monomial by a monomial; dividing a polynomial by a monomial; dividing a polynomial by a polynomial.

I. Dividing a Monomial by a Monomial

A. Suggested Procedure

1. Review with pupils that division of signed numbers is related to the operation of multiplication. Thus,

   $$15 \div 3, \text{ or } \frac{15}{3}$$

   means a number which when multiplied by 3 gives 15. Then,

   $$15 \div 3 = 5 \text{ because } 15 = 5 \cdot 3$$

   How can we develop the meaning of division in the set of polynomials to make it consistent with the meaning of division in the set of signed numbers?

2. Guide pupils to see that our definition of division in the set of polynomials should give us the same answer as in 1. Then, to be consistent, we will define division in the set of polynomials as the operation related to multiplication with polynomials.

   Thus, $\frac{x^3}{x^2} = x$ because $x^3 = x^2 \cdot x$

   $$\frac{x^7}{x^5} = x^2 \text{ because } x^7 = x^5 \cdot x^2$$

   $$\frac{x^{10}}{x^4} = x^6 \text{ because } x^{10} = x^4 \cdot x^6$$

3. Lead pupils to discover the law of exponents for division.

   a. In each of the above examples, have them compare the exponent of the quotient with the exponents of the dividend and divisor.

   $$\frac{x^3}{x^2} = x^{3-2} \qquad \frac{x^7}{x^5} = x^{7-5} \qquad \frac{x^{10}}{x^4} = x^{10-4}$$

b. Have pupils generalize that in finding the quotient of two powers having the same base, we write the base and use as its exponent the degree of the dividend minus the degree of the divisor. This is known as the law of exponents for division.

c. Have pupils understand the meaning of the 0th power of x.

$$\frac{x^3}{x^3} = 1 \quad \text{because } x^3 = x^3 \cdot 1$$

But if we apply the law of exponents for division

$$\frac{x^3}{x^3} = x^{3-3} = x^0$$

Then, if we wish to be consistent, we define $x^0 = 1$ for $x \neq 0$.

Note: The expression 0^0 is generally understood to have no meaning.

d. As a result of c, have pupils understand the meaning of ones place in the decimal system of numeration. For example, $7x^2 + 9x + 8$ may be thought of as $7x^2 + 9x^1 + 8x^0$. If $x = 10$, this becomes $7 \cdot 10^2 + 9 \cdot 10^1 + 8 \cdot 10^0$, or 798.

4. Have pupils understand how the quotient of two monomials whose coefficients are not 1 can be found.

a. Have them consider the following:

$$\frac{4x^5}{2x} = 2x^4 \quad \text{because } 4x^5 = 2x \cdot 2x^4$$

$$\frac{-8y^4}{2y^3} = -4y \quad \text{because } -8y^4 = 2y^3 \cdot (-4y)$$

$$\frac{3a^2b}{2a} = \frac{3}{2}ab \quad \text{because } 3a^2b = 2a \cdot (\tfrac{3}{2}ab)$$

Have pupils observe that in each of the above examples the coefficient of the quotient is the result of dividing the coefficients of the dividend and divisor.

b. Have pupils generalize that in dividing two monomials, we obtain the quotient by dividing the coefficients of the monomials and by using the law of exponents for division.

B. Suggested Practice

Perform the indicated divisions and check by multiplication. For which values of the variables are the following divisions meaningless? (When the variable is in the divisor, it may not be replaced by zero.)

1. $\frac{42a^2}{3}$
2. $72b^3 \div 4b$
3. $\frac{64a^2b^2}{8ab}$
4. $\frac{90xy^3}{10y}$
5. $\frac{-49m^3}{7m^2}$
6. $\frac{81r^2s^3}{-3r^2s}$
7. $\frac{108c^2d^2e^3}{-12ce^2}$
8. $\frac{-99x^5y^3z^2}{3x^2y^3z}$
9. $\frac{156pqr}{-6}$
10. $\frac{1}{2}gt^2 \div \frac{1}{2}t$
11. $-1.2d^5t^2 \div .3d^2t^2$
12. $\frac{1}{2}mv^2 \div \frac{1}{4}v^2$

II. Dividing a Polynomial by a Monomial

A. Suggested Procedure

1. Have pupils recall that division of a number by another number can be replaced by multiplication of the first number and the reciprocal of the divisor. For example,

$8 \div 2$ can be replaced by $8 \times \frac{1}{2}$

$10 \div 5$ can be replaced by $10 \times \frac{1}{5}$

$\frac{1}{2} \div 3$ can be replaced by $\frac{1}{2} \times \frac{1}{3}$ and, in general,

$a \div b$ can be replaced by $a \cdot \frac{1}{b}$ $\quad b \neq 0$

2. Pose puzzle problem:

Think of a whole number from 1 to 10. Square it, add the original number to this result. Now divide the sum by the original number. If you tell me your answer, I can tell you your original number.

Have several pupils state their answers. Tell them their original numbers. (The original number is 1 less than the answer.)

Have pupils see that the operations in the problem may be symbolized as follows:

x = original number

x^2 = square of the original number

$x^2 + x$ = original number added to the square

$\frac{x^2 + x}{x}$ = sum divided by the original number

How can the original number be determined?

Have pupils realize that what is involved is the division of a polynomial by a monomial.

3. Have pupils consider the division

$$(x^2 + x) \div x \text{ or } \frac{x^2 + x}{x}$$

Have them understand this division as follows:

$$\begin{aligned} \frac{x^2 + x}{x} &= (x^2 + x) \cdot \frac{1}{x} && \text{(Division may be replaced with multiplication by reciprocal)} \\ &= x^2 \cdot \frac{1}{x} + x \cdot \frac{1}{x} && \text{(Distributive property of multiplication over addition)} \\ &= \frac{x^2}{x} + \frac{x}{x} && \text{(Meaning of division)} \\ &= x + 1 && \text{(Division of monomials)} \end{aligned}$$

Observation: To divide a polynomial by a monomial, we make use of the distributive property of multiplication over addition.

4. After several such examples, have pupils note that we are, in effect, dividing each term of the polynomial by the monomial.

5. Have the pupils perform the following divisions and check by substitution: $a = 2$, $a = 3$, and $a = 0$.

a. $$\begin{aligned} \frac{30a^2 + 10a}{10} &= \frac{30a^2}{10} + \frac{10a}{10} \\ &= 3a^2 + a \end{aligned}$$

For which values of the variable is the division meaningful in this example? (for all values)

b. $$\frac{30a^2 + 10a}{10a} = \frac{30a^2}{10a} + \frac{10a}{10a}$$

$$= 3a + 1$$

For which values of the variable is the division meaningful in this example? (for all values except $a = 0$)

B. Suggested Practice

Perform the indicated divisions and check by multiplication. For which values of the variables are the divisions meaningless?

1. $\frac{12a + 15b}{3}$
2. $\frac{10m + 5n}{5}$
3. $\frac{25c + 5c}{c}$
4. $\frac{t^2 + 4t}{t}$
5. $\frac{3y^2 + 9y}{3y}$
6. $\frac{6r^3 + 9r^2 + 12r}{3r}$
7. $$\frac{m^2 - m}{-m} = \frac{m^2}{-m} + \frac{(-m)}{-m}$$

   $$= -m + 1$$

   or

   $$\frac{m^2 - m}{-m} = \frac{m^2}{-m} - \frac{m}{-m}$$

   $$= -m - (-1)$$

   $$= -m + 1$$
8. $\frac{8s - 4}{-4}$
9. $\frac{x^2 + 18x}{-x}$
10. $\frac{35c^3d + 7c^2d^2 - 14cd^3}{-7cd}$

III. Division of a Polynomial by a Polynomial

Note to teacher: Two approaches are suggested for developing division of a polynomial by a polynomial. The first approach emphasizes the similarity between the process of division for polynomials and long division for integers. The second approach stresses the use of the distributive property of multiplication over addition. Teachers will select the method which will best meet the needs of their pupils.

A. Suggested Procedure

1. Have pupils understand the meaning of the closure property of a set of numbers under an operation.

   a. Have them consider the result of adding two whole numbers (the positive integers and 0).

      $2 + 7 = 9$ $14 + 38 = 52$ $109 + 217 = 326$

      These are illustrations of the fact that the result of adding two whole numbers is always a whole number.

b. Have pupils consider the result of multiplying two whole numbers.

$2 \times 5 = 10$ $\qquad$ $0 \times 7 = 0$ $\qquad$ $21 \times 19 = 399$

These are illustrations of the fact that the result of multiplying two whole numbers is always a whole number.

c. Tell pupils that a set of numbers is closed under an operation if the result of the operation performed on every pair of elements in the set is also an element of the set. Thus, the set of whole numbers is closed under addition and under multiplication.

d. Have pupils realize that a set need not be closed with respect to an operation.

   1) The set of whole numbers is not closed under subtraction. For example, $4-6$ does not yield a whole number.

   2) The set of whole numbers is not closed under division. For example, $3 \div 4$ does not yield a whole number.

   3) The set of odd numbers is not closed under addition since the sum of two odd numbers is an even number (not in the set).

   4) Guide pupils to see that to make subtraction always possible, the set of arithmetic numbers was extended to form the set of signed numbers; to make division always possible, the set of whole numbers was extended to form the set of fractions.

2. Have pupils review the meaning of a polynomial and the degree of a polynomial.

3. Guide them to realize that the set of polynomials is not closed under division. For example, if the polynomial $x^2 + 2x + 1$ is divided by the polynomial x, we have

$$\frac{x^2 + 2x + 1}{x} = \frac{x^2}{x} + \frac{2x}{x} + \frac{1}{x}$$

$$= x + 2 + \frac{1}{x} \quad \text{This is not a polynomial.}$$

Note: Just as in the past we extended the set of whole numbers to form the set of arithmetic fractions in order to make division of numbers always possible, so will the set of polynomials be extended to form the set of algebraic fractions (rational expressions), in order that division of polynomials will always be possible. (See Chapter X)

4. Pose problem: The area of a rectangle is $x^2 + 3x + 2$. Its width is $x + 1$. What is its length? Have pupils see that what is involved is the division of polynomials.

$$(x^2 + 3x + 2) \div (x + 1)$$

a. Have pupils compare the division of polynomials with the division of integers.

1) Have them consider: $132 \div 11$

$$\begin{array}{r} 12 \\ 11\overline{)\,132} \\ \underline{110} \\ 22 \\ \underline{22} \end{array} \qquad \begin{array}{r} 10 + 2 \\ 10+1\overline{)\,100 + 30 + 2} \\ \underline{100 + 10}\phantom{{}+2} \\ 20 + 2 \\ \underline{20 + 2} \end{array}$$

Have pupils see that the dividend and divisor may be written in terms of powers of 10.

$$\begin{array}{r} 10 + 2\phantom{{}+2} \\ 10+1\overline{)\,10^2 + 3\cdot10 + 2} \\ \underline{10^2 + 1\cdot10}\phantom{{}+2} \\ 2\cdot10 + 2 \\ \underline{2\cdot10 + 2} \end{array}$$

If x is used to represent the base 10, the problem becomes:

$$\begin{array}{r} x + 2\phantom{{}+2} \\ x+1\overline{)\,x^2 + 3x + 2} \\ \underline{x^2 + x}\phantom{{}+2} \\ 2x + 2 \\ \underline{2x + 2} \\ 0 \end{array}$$

Have pupils see that when $x^2 + 3x + 2$ is divided by $x + 1$, the answer is the polynomial $x + 2$.

Note: While this procedure was developed by means of a consideration of division in base ten, it is equally valid in any other base, and, in fact, the variable x may represent any number.

2) Have pupils check the division by multiplication.

Have pupils check the division by replacing x by any convenient number. What numbers, if any, are not permissible as replacements for x? (-1 is not permissible since it leads to a zero divisor.)

3) After several similar examples, have pupils consider a division problem with a remainder other than zero.

$$\begin{array}{rl}
 & 3x + 2 \\
x + 4) & \overline{3x^2 + 14x + 10} \\
 & \underline{3x^2 + 12x} \\
 & \qquad 2x + 10 \\
 & \qquad \underline{2x + 8}
\end{array}$$

The quotient is $3x + 2$ with a remainder of 2. Guide pupils to express the remainder as a fractional part of the divisor:

$$(3x^2 + 14x + 10) \div (x + 4) = 3x + 2 + \frac{2}{x+4}$$

Have pupils note that $3x + 2 + \frac{2}{x+4}$ is not a polynomial.

b. Alternate approach. Have pupils understand the division of polynomials as follows:

1) When the polynomial $x^2 + 3x + 2$ is divided by the polynomial $x + 1$, the polynomial answer, if it exists, must be of degree one, since it must multiply a first-degree polynomial $(x+1)$ to give a second degree polynomial $(x^2 + 3x + 2)$ as a product. Thus,

$$(x + 1)\cdot(\_\_ + \_\_) = x^2 + 3x + 2$$

What must the first term of the answer be?

a) Guide pupils to realize that the first term of the answer is x (that is, $x^2 \div x$).

Then, $(x + 1)\cdot(x + \_\_) = x^2 + 3x + 2$

b) Have pupils use the distributive property of multiplication over addition to expand the left side.

$$(x + 1)\cdot x + (x + 1)\ (\_\_) = x^2 + 3x + 2$$
$$x^2 + x + (x + 1)\cdot(\_\_) = x^2 + 3x + 2$$

c) Have pupils add the additive inverse of $x^2 + x$ to both sides of the equation. Then,

$$(x + 1)\cdot(\_\_) = x^2 + 3x + 2 - x^2 - x$$
$$(x + 1)\cdot(\_\_) = 2x + 2$$

What must the second term of the answer be?

d) Guide pupils to see that if $(x + 1)\cdot(\ \underline{\quad}\ )$ is to give $2x + 2$ as a result, the second term of the answer must be 2 (that is, $2x \div x$).

e) $(x + 1)(2) = 2x + 2$

Therefore, when $x^2 + 3x + 2$ is divided by $x + 1$, the answer is the polynomial $x + 2$.

Check by multiplication:

$$\begin{array}{ll}
(x + 1)(x + 2) & \stackrel{?}{=} x^2 + 3x + 2 \\
(x + 1)x + (x + 1)2 & \stackrel{?}{=} x^2 + 3x + 2 \\
x^2 + x + 2x + 2 & \stackrel{?}{=} x^2 + 3x + 2 \\
x^2 + 3x + 2 & = x^2 + 3x + 2 \text{ and it checks}
\end{array}$$

2) After several similar examples, have pupils see that the work may be arranged in the following short form:

$$\begin{array}{rrl}
 & x & \text{see a)} \\
x+1) & \overline{x^2 + 3x + 2} & \\
 & \underline{x^2 + x} & \text{see b)} \\
 & 2x + 2 & \text{see c)}
\end{array}
\qquad\qquad
\begin{array}{rrl}
 & x + 2 & \text{see d)} \\
x+1) & \overline{x^2 + 3x + 2} & \\
 & \underline{x^2 + x} & \\
 & 2x + 2 & \\
 & \underline{2x + 2} & \text{see e)}
\end{array}$$

3) Have pupils consider the following problem: Can the polynomial $3x^2 + 14x + 10$ be divided by the polynomial $x + 4$ to give a polynomial as an answer?

If $(x + 4)(\ \underline{\quad}\ + \ \underline{\quad}\ ) = 3x^2 + 14x + 10$, then the first term of the answer is $3x$ (that is, $3x^2 \div x$).

$$\begin{array}{rl}
\text{Then, } (x + 4)(3x + \underline{\quad}) & = 3x^2 + 14x + 10 \text{ or} \\
(x + 4)\cdot 3x + (x + 4)(\ \underline{\quad}\ ) & = 3x^2 + 14x + 10 \text{ or} \\
3x^2 + 12x + (x + 4)(\ \underline{\quad}\ ) & = 3x^2 + 14x + 10 \text{ and} \\
(x + 4)(\ \underline{\quad}\ ) & = 2x + 10
\end{array}$$

If $(x + 4)(\underline{\quad\quad}) = 2x + 10$, then the second term of the answer appears to be 2 (that is, $2x \div x$).

But $(x + 4)(3x + 2) = 3x^2 + 14x + 8 \neq 3x^2 + 14x + 10$.

Have pupils conclude that $3x^2 + 14x + 10$ is not divisible by the polynomial $x + 4$ to give an answer that is a polynomial.

Have them note that $3x^2 + 14x + 10 = (x + 4)(3x + 2) + 2$. Have them realize that 2 is a remainder when dividing $3x^2 + 14x + 10$ by $x + 4$.

The above division example can be arranged in short form as follows:

$$
\begin{array}{rl}
 & \underline{3x \;+\; 2} \\
x+4) & 3x^2 + 14x + 10 \\
 & \underline{3x^2 + 12x} \\
 & 2x + 10 \\
 & \underline{2x + 8} \\
 & 2
\end{array}
$$

The quotient is $3x + 2$ with a remainder of 2. This can be written as:

$$3x + 2 + \frac{2}{x+4}$$

5. Have pupils see the need for using the descending (or ascending) order of the dividend and divisor to facilitate the mechanics of division of a polynomial by a polynomial.

6. Guide pupils to see how missing terms in a dividend are provided for, by using zero as a coefficient.

   Example: Divide $x^3 - x + 6$ by $x + 2$

$$
\begin{array}{rl}
 & \underline{x^2 - 2x \;\; + 3} \\
x+2) & x^3 + 0 \cdot x^2 - x + 6 \\
 & \underline{x^3 + 2x^2} \\
 & - 2x^2 - x \\
 & \underline{- 2x^2 - 4x} \\
 & 3x + 6 \\
 & \underline{3x + 6}
\end{array}
$$

   Check by multiplication.

7. Have pupils practice division of polynomials and summarize the procedure. Have them check division by multiplication. It may also be checked by numerical substitution.

B. Suggested Practice

Perform the following divisions and check your answers.

1. $\dfrac{x^2 + 7x + 10}{x + 2}$
2. $\dfrac{y^2 - 3y + 2}{y - 1}$
3. $\dfrac{w^2 - w - 72}{w + 8}$
4. $\dfrac{8c^2 + 16cd + 6d^2}{2c + 3d}$
5. $(x^2 + 3x + 5) \div (x - 2)$
6. $p-2\overline{)\,3p^2 - 7p + 4}$
7. $(-2 - a + 3a^2) \div (a - 1)$
8. $x+2\overline{)\,3x^3 + 6x^2 - x - 2}$
9. $(y^3 - y + 6) \div (y + 2)$
10. $x-3\overline{)\,x^3 + 27}$

11. If one factor of $2m^2 - mn - 6n^2$ is $m - 2n$, what is the other factor?

12. The area of a rectangle is $x^2 - x - 12$, and its length is $x - 4$. What is its width?

13. The volume of a rectangular sold is $m^3 + m^2 + m + 6$. Its height is $m + 2$. Find the area of the base.

14. See various textbooks for additional practice.

CHAPTER IX

SPECIAL PRODUCTS AND FACTORING

This section presents materials and procedures for helping pupils develop understanding and skill in factoring algebraic expressions through the use of the basic number properties.

I. Greatest Common Factor of Monomials

A. Suggested Procedure

1. Review meaning of factor

a. Pose problem: The area of a rectangle is 24. What are the dimensions if they are to be whole numbers?

Pupils will suggest various possibilities:

6 and 4 because $6 \times 4 = 24$
8 and 3 because $8 \times 3 = 24$
12 and 2 because $12 \times 2 = 24$
24 and 1 because $24 \times 1 = 24$

b. Have them recall that when two or more numbers are multiplied to give a product, each number is a factor of the product. Thus, 6, 4, 8, 3, 12, 2, 24, 1 are all factors of 24.

Elicit that when a product and one factor are known, we may find the other factor by dividing the product by the known factor.

Note: The factors of an integer are also called the divisors of the integer.

2. Have pupils express a number such as 12 as the product of numbers (positive or negative) in various ways:

$12 = 1 \cdot 12$
$12 = 2 \cdot 6$
$12 = (-2)(-6)$
$12 = 2 \cdot 3 \cdot 2$

$12 = \frac{1}{2} \cdot 24$
$12 = 18 \cdot \frac{1}{3} \cdot 6$
$12 = (-\frac{1}{4})(-24) \cdot 2$ etc.

Have pupils realize that if fractions were considered as factors of an integer, the number of factors of this integer would be limitless, since any number ($\neq 0$) would be a factor. It is therefore customary to restrict the factors of a number to a given set of integers. (Unless otherwise stated, the factors of a polynomial will have coefficients that are in the same set of numbers as the coefficients of the given polynomial.)

Have pupils conclude that to factor an integer is to find several integers (positive or negative) whose product is the given number.

3. Have pupils consider the following subset of the set of integers:

   2, 3, 5, 7, 11, 13, 17, ...

   What common properties do these have? Each number in this set is greater than one and each number has only itself and one as positive integral factors. When an integer greater than one has only itself and one as positive integral factors, it is called a prime number.

   Have pupils list several other members of the set of prime numbers. Have them list the set of all even primes.

   Have them consider whether 179 is a prime number. Does it have any factors other than 1 and 179? Try 2, 3, 5, 7, 9, 11, 13 as possible divisors or factors. Why do we not use even numbers greater than 2? Why do we not use any number greater than 13? ($14^2 = 196$ which is greater than 179. Therefore, if there were a factor greater than 13, there would necessarily be another factor less than 13.)

4. Have pupils practice expressing positive integers as the product of prime numbers.

   a. $18 = 2 \cdot 9 = 2 \cdot 3 \cdot 3$ or
      $18 = 3 \cdot 6 = 3 \cdot 2 \cdot 3$

   b. $24 = 12 \cdot 2 = 4 \cdot 3 \cdot 2 = 2 \cdot 2 \cdot 3 \cdot 2$ or
      $24 = 8 \cdot 3 = 2 \cdot 2 \cdot 2 \cdot 3$ or
      $24 = 6 \cdot 4 = 3 \cdot 2 \cdot 2 \cdot 2$

      Have them notice that in each case the prime factors are the same except for the order in which they appear. Inform pupils that it can be proven that any natural number can be written as the product of a unique set of prime factors. Thus, we would not say that $2 \cdot 3 \cdot 3$ and $3 \cdot 2 \cdot 3$ are different factorizations of 18 ($18 = 2 \cdot 3^2$).

5. Have pupils practice expressing negative integers as the product of -1 and prime numbers.

   a. $-20 = -1 \cdot 5 \cdot 2 \cdot 2 = -1 \cdot 5 \cdot 2^2$

   b. $-45 = -1 \cdot 3 \cdot 3 \cdot 5 = -1 \cdot 3^2 \cdot 5$

6. Help pupils to find a common factor of a pair of monomials.

   a. What is the greatest common factor of 120 and 36? We can determine this by factoring these numbers into the product of primes because we know that each has a unique set of prime factors.

$$\begin{aligned} 120 &= 2\cdot 60 \\ &= 2\cdot 2\cdot 30 \\ &= 2\cdot 2\cdot 2\cdot 15 \\ &= 2\cdot 2\cdot 2\cdot 3\cdot 5 \\ &= 2^3\cdot 3\cdot 5 \end{aligned}$$

$$\begin{aligned} 36 &= 2\cdot 18 \\ &= 2\cdot 2\cdot 9 \\ &= 2\cdot 2\cdot 3\cdot 3 \\ &= 2^2\cdot 3^2 \end{aligned}$$

   The largest power of 2 common to 120 and 36 is 2^2. The largest power of 3 common to both numbers is 3. There are no other factors in common. Then the greatest common factor is $2^2\cdot 3$ or 12.

   b. Have pupils find the greatest common factor of a pair of monomials, $27a^2b$ and $36a^3b$.

   Have pupils express each numerical coefficient as a product of prime integers:

   $3^3\cdot a^2\cdot b$ and $2^2\cdot 3^2\cdot a^3\cdot b$

   What is the largest number that is a factor of 27 a^2b and 36 a^3b? (3^2 or 9)
   What is the highest power of a that is a factor of each? (a^2)
   What is the highest power of b that is a factor of each? (b)
   Then what is the greatest common factor of $27a^2b$ and $36a^3b$? ($9a^2b$)

   c. Have pupils find the greatest common factor of this pair of monomials:

   $25x^2y^3$ and $-50x^2y$

   Expressing each coefficient as the product of prime factors, we have $5^2x^2y^3$ and $(-1)\cdot 2\cdot 5^2x^2y$

   5^2x^2y or $25x^2y$ is the greatest common factor of both.

B. Suggested Practice

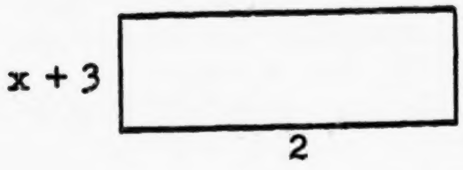
Find the greatest common factor of each of the following pairs:

1. 12,32
2. 50,75
3. 4a, 8
4. $8b^2$, 6b
5. $9rs^2$, $6r^2$
6. $6x^2y$, $-12x^3y^2$
7. $5a^2b^3$, $-10a^2b$
8. 42xyz, $28x^2y^2$

II. Finding the Greatest Common Monomial Factor of the Terms of a Polynomial

A. Suggested Procedure

1. Have pupils find the area of a rectangle whose dimensions are 2 and $x + 3$.



$A = bh$
$A = 2(x + 3)$
$A = 2(x)+2(3)$ Distributive property
$A = 2x + 6$

Since the product, $2x + 6$, was obtained by multiplying 2 and $x + 3$, then 2 and $x + 3$ are the factors of $2x + 6$. If $2x + 6$ were given as the product of two factors, how would we determine those factors? Have pupils see that the binomial $2x + 6$ consists of two terms, $2x$ and 6. Factors of $2x$ are 2 and x. Factors of 6 are 2 and 3. Because 2 is a factor of $2x$ and of 6, it is called the common factor of $2x$ and 6.

Then, $2x + 6 = 2\cdot x + 2\cdot 3$ and, using the distributive property, we can write this as

$$2x + 6 = 2(x + 3)$$

This sentence is true for any replacement of the variable by a signed number. The given polynomial has 2 and $x + 3$ as factors.

2. In a similar way, have pupils show that

$$6x^2 - 9x = 3x (2x - 3)$$

Have them note that $3x$ is a monomial factor of the polynomial $6x^2 - 9x$. If a monomial is a factor of every term of a polynomial, it is called a common monomial factor of the polynomial.

3. Have pupils consider various ways of expressing $8x^2 - 12x$ as the product of factors, one of which is a monomial:

$$8x^2 - 12x = 2(4x^2 - 6x)$$
$$8x^2 - 12x = 4(2x^2 - 3x)$$
$$8x^2 - 12x = 2x(4x - 6)$$
$$8x^2 - 12x = 4x(2x - 3)$$

Have them see that $4x$ is the greatest common monomial factor since, of all the common factors, it has the greatest numerical coefficient and is of the greatest degree. In expressing a polynomial as the product of factors, one of which is a monomial factor, it is understood that we use the greatest common monomial factor.

4. Have pupils complete the table below. Have them check by multiplication, or by substitution.

	Polynomial	GCF	$\frac{\text{Polynomial}}{\text{GCF}}$ = Quotient	Factored Form
a.	$5x^2 - 10$	5	$\frac{5x^2}{5} - \frac{10}{5} = x^2 - 2$	$5(x^2 - 2)$
b.	$9a^2x - 18ax^2$	$9ax$	$\frac{9a^2x}{9ax} - \frac{18ax^2}{9ax} = a - 2x$	$9ax(a-2x)$
c.	$12a^3b + 16a^2b^2$			
d.	$\pi r^2 + \pi rl$			
e.	$ab + b$			

B. Suggested Practice

1. Write in factored form:

   a. $2a + 2b$
   b. $5x - 10y$
   c. $3x^2 + 6x^3$
   d. $4b - 8b^2$
   e. $10x^2y + 15xy^2$
   f. $a + a^2 + a^3$
   g. $u^2v - uv^2$
   h. $7x^5 + 21x^3 - 56x$
   i. $9a^3b^2 + 18a^2b^2 - 6a^2b^3$
   j. $-t^3 + 4t^2 + 8t$

2. Select additional examples from textbooks.

3. Write each expression in factored form. (OPTIONAL)

   a. $x(x+1) + 3(x+1)$ Solution: $(x+1)(x+3)$
   b. $y(y-5) + 10(y-5)$
   c. $(m-n)r^2 + (m-n)t^2$
   d. $a^2 - a + ab - b$

III. Squaring a Binomial; Factoring Trinomial Squares

A. Suggested Procedure

Note to Teacher: If preferred, the general case of factoring trinomial products may be developed first. Following this, the factoring of trinomial squares and the factoring of the difference of two squares may be taught as special cases.

1. Have pupils review the multiplication of a binomial by a binomial.

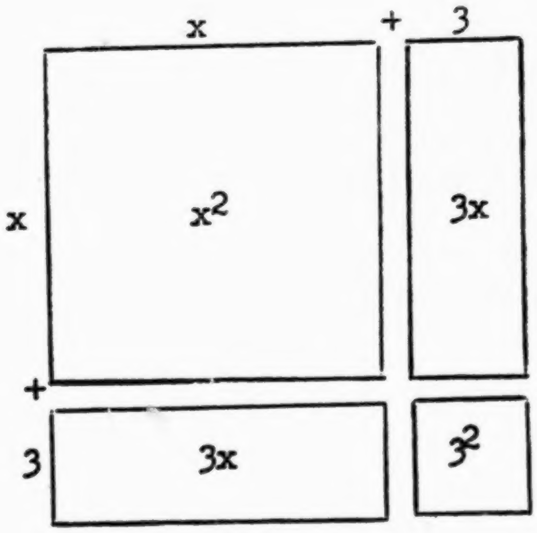
   Refer to Chapter V for the use of the distributive property in multiplying polynomials.

2. Have pupils consider the special case of multiplying a binomial by itself.

   a. $(x + 3)^2 = (x + 3)(x+3)$

$$\begin{aligned} &= (x + 3)x + (x + 3)3 \\ &= x^2 + 3x + 3x + 9 \\ &= x^2 + 6x + 9 \end{aligned}$$

   Have pupils note that the area of the square whose side is $(x + 3)$ is made up of the areas of 2 squares and 2 identical rectangles.



$x^2 + 2(3x) + 3^2$

b. Have pupils perform several multiplications of identical binomials using the usual method of multiplication.

Factored Form		Product
$(y + 5)(y + 5)$ or $(y + 5)^2$	=	$y^2 + 10y + 25$
$(a - 2)(a - 2)$ or $(a - 2)^2$	=	$a^2 - 4a + 4$
$(b + 1)(b + 1)$ or $(b + 1)^2$	=	$b^2 + 2b + 1$

Pose questions:

1) What kind of polynomial is the product? (a trinomial)

2) How is the first term of the product related to the first term of the binomial in each case? (It is the square of the first term of the binomial.)

3) How would you describe the last term of each product? (It is always positive. It is the square of the second term of the binomial.)

4) How would you describe the second term of each product? (It is twice the product of the terms of the binomial)

3. Have pupils use the observed pattern to work the following examples mentally.

$(x + 2)(x + 2)$ $(t + 7)^2$ $(x + \frac{1}{2})^2$

$(y - 4)^2$ $(r - 10)^2$ $(2 + a)(2 + a)$

Tell pupils that a trinomial which is the square of a binomial is called a trinomial square.

4. Have pupils factor trinomial squares.

a. Pose question: How can we recognize a trinomial that is the square of a binomial? For example, is $x^2 + 12x + 36$ a trinomial square?

Have pupils recall the relationship between the terms of the trinomial square and the terms of the binomial.

Is the first term of the trinomial a square? Yes, x^2 is the square of x.

Is the last term a square? Yes, 36 is 6^2.

Is the middle term twice the product of x and 6? Yes, $12x = 2(x)(6)$.

Have pupils conclude that the trinomial is the square of a binomial.

b. Lead pupils to see that the trinomial $x^2 + 12x + 36$ may then be written in factored form as:

$$x^2 + 12x + 36 = (x + 6)(x + 6) \text{ or } (x + 6)^2$$

Have them check factoring by multiplication.

c. In similar fashion, have pupils see that $a^2 - 14a + 49$ is a trinomial square. Have them observe that:

a^2 is the square of a $\qquad a^2 = (a)^2$

49 is the square of -7 $\qquad 49 = (-7)^2$

$-14a$ is twice the product of a and -7 $\qquad -14a = 2(a)(-7)$

Then, $a^2 - 14a + 49 = (a - 7)(a - 7)$ or $(a - 7)^2$

d. Have pupils consider whether $a^2 - 14a - 49$ is a trinomial square.

Is -49 the square of a term? Since $(+7)^2 = +49$ and $(-7)^2 = +49$, then -49 is not the square of a term and $a^2 - 14a - 49$ cannot be a trinomial square.

B. Suggested Practice

1. Which of the following are trinomial squares? Explain.

   a. $x^2 + 6x + 9$
   b. $x^2 - 8x + 16$
   c. $a^2 + 4a + 2$
   d. $y^2 + 10y + 25$
   e. $b^2 - 2b - 1$

2. Factor, if possible, and check by multiplication.

   a. $x^2 + 8x + 16$
   b. $y^2 - 10y + 25$
   c. $b^2 + 18b + 81$
   d. $1 + 2a + a^2$
   e. $x^2 + 2xy + y^2$
   f. $t^2 + 2t - 1$ (not possible)
   g. $16 - 8b + b^2$
   h. $x^2 - 2xy + y^2$

OPTIONAL

3. Factor, if possible, and check by multiplication.

a. $4x^2 - 4x + 1$	c. $9r^2 - 6rt + t^2$
b. $4x^2 + 12xy + 9y^2$	d. $25y^2 - 10y - 1$

4. What must be the value of m, if each trinomial is to be a square?

a. $y^2 + my + 16$	$(m = +8 \text{ or } -8)$
b. $a^2 + 6x + m$	$(m = 9)$
c. $my^2 - 10y + 1$	$(m = 25)$

IV. Multiplying Two Binomials Whose Product Is a Binomial; Factoring the Difference of Two Squares

A. Suggested Procedure

1. Have pupils recall that the product of two binomials is usually a polynomial with three or four terms.

   Pose question: Can two binomials be such that their product is a binomial?

   a. Have pupils multiply two identical binomials, such as

$$\begin{aligned}(x + 3)(x + 3) &= (x + 3)x + (x + 3)x \\ &= x^2 + 3x + 3x + 9 \\ &= x^2 + 6x + 9 \qquad \text{The product is a trinomial.}\end{aligned}$$

   1) Have them observe that the product contains the square of x, the first term of each binomial, and the square of 3, the second term. Elicit that if the product is to be a binomial, the middle term will be lacking.

   2) Guide pupils' thinking as follows:

   In the above multiplication, the middle term, 6x, was obtained by adding 3x and 3x. What must be true of these monomials if their sum is to be 0x instead of 6x? (They must be additive inverses.) Then, instead of x + 3 and x + 3, what two binomials shall we use? (x + 3 and x - 3)

3) Have pupils verify by multiplication that $(x + 3)(x - 3)$ results in a binomial product.

b. Have pupils suggest several other examples of multiplication of binomials which result in a binomial product.

$(x + 5)(x - 5) = x^2 - 25$

$(y + 2)(y - 2) = y^2 - 4$

$(-x + 7)(-x - 7) = x^2 - 49$

c. Have pupils realize that if in two binomials, the first terms are the same and the second terms are additive inverses of each other, then the product of the binomials is also a binomial. Have them note that the product in each case can be described as the difference of two squares of monomials.

d. Have pupils express the following products as binomials.

1) $(x + 8)(x - 8)$

2) $(y - 9)(y + 9)$

3) $(c + .2)(c - .2)$

4) $(a + 6b)(a - 6b)$

5) $(y + 2x)(2x - y)$

6) $(x^2 + 4)(x^2 - 4)$

7) $(a + \frac{1}{4})(a - \frac{1}{4})$

8) $(R + r)(R - r)$

9) $(47)(53)$ or $(50 - 3)(50 + 3)$

10) $(62)(58)$

11) $(36)(44)$

12) 25×35

2. Have pupils learn to factor a difference of two squares.

a. Have them recall that the factors of a polynomial have coefficients that are in the same set of numbers as the coefficients of the polynomial. Thus, if the coefficients of a polynomial are integers, its factors will have integral coefficients.

b. Have pupils determine which of the following can be expressed as the product of two equal factors. Have them then write the expression as the square of a monomial.

1) $x^2 = x \cdot x$ or $(x)^2$

2) $4x^2 = 2x \cdot 2x$ or $(2x)^2$

3) $m^4 = m^2 \cdot m^2$

4) $9y^4 = 3y^2 \cdot 3y^2$ or $(3y^2)^2$

5) $.49a^2 = (.7a)(.7a)$ or $(.7a)^2$

6) $3x^2$ Cannot be expressed as the product of two equal factors since 3 does not have two equal integral factors. Then $3x^2$ is not the square of a monomial.

c. Have pupils express, if possible, each of these binomials as the difference of two squares of monomials as follows:

1) $x^2 - 9 = (x)^2 - (3)^2$ (The monomials are x and 3)

2) $4x^2 - 25 = (2x)^2 - (5)^2$ (The monomials are 2x and 5)

3) $y^2 - 4x^2 = (y)^2 - (2x)^2$ (What are the monomials?)

4) $a^2 - \frac{1}{4} = (a)^2 - (\frac{1}{2})^2$ (What are the monomials?)

5) $b^2 - 5$ (Cannot be so expressed. Why not?)

6) $d^2 - .09 = (d)^2 - (.3)^2$ (What are the monomials?)

7) $x^2 + 4$ (Cannot be so expressed. Why not?)

8) $3x^2 - 4y^2$ (Cannot be so expressed. Why not?)

d. Have pupils realize that since the product of two binomials whose first terms are the same, and whose second terms are additive inverses results in a binomial which is the difference of two squares of monomials, we may reverse the process to factor the difference of two squares.

We factor the difference of two squares of monomials into the sum and difference of the monomials. For example,

$$x^2 - 9 = (x)^2 - (3)^2 = (x + 3)(x - 3)$$

$$4x^2 - 25 = (2x)^2 - (5)^2 = (2x + 5)(2x - 5)$$

$$y^2 - 4x^2 = (y)^2 - (2x)^2 = (y + 2x)(y - 2x)$$

$$x^2 - a^2 = (x)^2 - (a)^2 = (x + 2)(x - a) \text{ etc.}$$

B. Suggested Practice

1. Factor the following differences of squares.

a. $c^2 - d^2$

b. $4a^2 - 9b^2$

c. $x^2 - .04$

d. $9x^2 - y^2$

e. $R^2 - r^2$

f. $x^2 - 16y^2$

g. $16m^2 - 25n^2$

h. $\frac{1}{4}x^2 - y^2$

i. $.09y^2 - z^2$

j. $49y^2 - 25x^2$

k. $\frac{1}{9}a^2 - \frac{1}{25}b^2$

2. Factor each of the following integers.

a. $15^2 - 1$ Solution: $15^2 - 1 = (15 + 1)(15 - 1) = (16)(14)$

b. 2491 Solution:
$$\begin{aligned} 2491 &= 2500 - 9 \\ &= (50 - 3)(50 + 3) \\ &= (47)(53) \end{aligned}$$

c. 3596

d. 4875

V. Finding Trinomial Products by Inspection; Factoring Simple Trinomials (General Case)

A. Suggested Procedure

1. Have pupils use the distributive property to find the product of the following pair of binomials:

$$\begin{aligned} (x + 2)(x + 5) &= (x + 2)x + (x + 2)5 \\ &= x^2 + 2x + 5x + 10 \\ &= x^2 + (2 + 5)x + 10 \\ &= x^2 + 7x + 10 \end{aligned}$$

Have them observe the following for this product:

a. The first term, x^2, of the product is obtained by multiplying the first terms of the binomials.

$(x + 2)(x + 5)$ $x \cdot x = x^2$

b. The middle term, $7x$, of the product is obtained by multiplying the first term of each binomial by the second term of the other and adding these products.

$(x + 2)(x + 5)$ $2x + 5x = (2 + 5)x = + 7x$

c. The last term, 10, of the product is obtained by multiplying the two last terms of the binomials.

$(x + 2)(x + 5)$ $2 \cdot 5 = +10$

2. Present additional illustrations and have pupils make these observations again.

   Have pupils use these observations to complete the following table:

Binomial Factors			Product	
		First Term	Middle Term	Last Term
$(x + 3)(x + 5)$	=	?	$+8x$	?
$(y - 2)(y + 1)$	=	?	$-y$	?
$(a + 3)(a - 7)$	=	?	$-4a$	?
$(r - 6)(r - 2)$	=	?	$-8r$	?
$(b + 2)(b + 8)$	=	b^2	?	$+16$
$(p - 9)(p - 11)$	=	p^2	?	$+99$
$(x - 5)(x + 7)$	=	x^2	?	-35
$(b + 2)(b - 3)$	=	b^2	?	-6
$(b + 4)(b + 4)$	=	?	?	?
$(y - 4)(y - 5)$	=	?	?	?
$(a - 2)(a + 8)$	=	?	?	?
$(m + 7)(m - 11)$	=	?	?	?

3. Have pupils learn to factor simple trinomials which are the product of two binomials.

   a. Have pupils multiply $(x+2)$ by $(x+3)$. The product is $x^2 + 5x + 6$.

   b. Pose problem: The area of a rectangle is represented by $x^2 + 5x + 6$.

      What binomial expressions may represent the dimensions?

      Elicit that the binomial expressions are factors of the product $x^2 + 5x + 6$.

      Guide pupils' thinking as follows:

      1) What is the first term of each binomial? The first term of each is x, since the product of x and x is x^2, the first term of the trinomial.

$$x^2 + 5x + 6 = (x + ?)(x + ?)$$

      2) Since 6 was obtained by multiplying the last terms of the binomials, these last terms must be factors of 6. What are all the possible pairs of factors of 6? They are:

         1 and 6
         2 and 3
         -1 and -6
         -2 and -3

3) Which pair must be chosen as last terms, so that the product of two binomials will have a middle term of $5x$? (2 and 3)

4) Then we can write the factors as follows:

$$x^2 + 5x + 6 = (x + 2)(x + 3)$$

Have pupils check factoring by multiplication.

Is the form $x^2 + 5x + 6 = (x + 3)(x + 2)$ acceptable? Why?

The length of the rectangle may be represented by $x + 3$; the width by $x + 2$.

c. Have pupils factor the following:

$x^2 + 3x + 2$	$(x + 1)(x + 2)$
$x^2 + 4x + 3$	$(x + 1)(x + 3)$
$x^2 + 7x + 10$	$(x + 2)(x + 5)$

Have them note the sign of the constant term, the patterns of the signs in the factors, and the sign of the coefficient of the middle terms.

d. Have pupils factor the following:

1) $x^2 - 4x + 3$

We can write $x^2 - 4x + 3 = (x + ?)(x + ?)$.

What are the possible pairs of factors of $+3$? (1 and 3, -1 and -3).

Which pair must be chosen as last terms so that the product of the two binomials will have a middle term of $-4x$? (-1 and -3)

Therefore, $x^2 - 4x + 3 = (x - 3)(x - 1)$.

Check the factors by multiplying.

2) $x^2 - 3x + 2$	$(x - 2)(x - 1)$
3) $y^2 - 8y + 7$	$(y - 1)(y - 7)$
4) $x^2 - 7x + 10$	$(x - 5)(x - 2)$

Have them note the sign of the constant term, the patterns of the signs in the factors, and the sign of the coefficient of the middle terms.

e. Have pupils factor the following:

1) $x^2 - 2x - 8$

The product of the numbers which are needed as the second terms in the binomials $(x + ?)$ and $(x + ?)$ is -8.

What are the possible pairs of factors of -8? (-8 and 1; 8 and -1; 4 and -2; -4 and 2)

Which pair must be chosen as last terms so that the product of the binomials will have a middle term of $-2x$? (-4 and 2)

$x^2 - 2x - 8 = (x - 4)(x + 2)$ Check by multiplication.

2) $x^2 - 3x - 4$ $(x - 4)(x + 1)$

3) $x^2 + 3x - 4$ $(x + 4)(x - 1)$

4) $x^2 - 5x - 6$ $(x - 6)(x + 1)$

Have them note the sign of the constant term, the patterns of the signs in the factors, and the sign of the coefficient of the middle terms.

B. Suggested Practice

1. $x^2 + 5x + 6 = (x\ ?\ 2)(x\ ?\ 3)$
2. $x^2 - 5x + 4 = (x\ ?\ 4)(x\ ?\ 1)$
3. $x^2 - 5x - 6 = (x - 6)(x + ?)$
4. $x^2 + 3x - 4 = (x + ?)(x - ?)$
5. $y^2 + 5y - 6 = ?$
6. $y^2 - 6y + 8 = ?$
7. $c^2 + 8c - 9 = ?$
8. $b^2 - 8b - 9 = ?$
9. $d^2 + 6d + 9 = ?$
10. $y^2 + 4y + 4 = ?$
11. $a^2 - a - 2 = ?$
12. The area of a rectangle is $x^2 - x - 6$. What are the sides of the rectangle if they are factors of the area?
13. The area of a rectangle is $y^2 + 2y - 15$. What may its base and altitude be?

OPTIONAL

14. If $x^2 + bx + c$ is factorable into $(x + r)(x + s)$, what is true of the signs of r and s, if c is a positive number?

    What is true of the signs of r and s if c is a negative number?

    If c is a positive number, how does the sign of b affect the signs of r and s?

    If c is a negative number, how is the sign of b related to the absolute values of r and s?

VI. Factoring Trinomials of the Form $ax^2 + bx + c$, where $a = 2$ or 3

A. Suggested Procedure

1. Review with pupils the relationship between the signs of the terms of a trinomial product and the signs of the terms of the binomial factors.

2. Pose problem: The area of a rectangle is $3x^2 + 7x + 2$. What binomial factors might represent the dimensions?

   Guide pupils' thinking as follows:

   a. If the first term of the trinomial product is $3x^2$, what are the first terms of the binomial factors? ($3x$ and x) Thus, we may write:

   $$3x^2 + 7x + 2 = (3x + ?)(x + ?)$$

   Note: The first term of each factor should have a positive coefficient.

   b. What is the product of the second terms of the binomials? ($+2$)

   What are possible pairs of factors of $+2$? (2 and 1, -2 and -1)

   Which factors do we reject? Why?

   c. How may we fill in the blanks in the binomial factors?

   1) Have pupils try $(3x + 2)(x + 1)$. Have them check the result by multiplication to see whether the factoring is correct.

   $$(3x + 2)(x + 1) = 3x^2 + 5x + 2 \quad \text{It does not check.}$$

   2) Have pupils try $(3x + 1)(x + 2)$.

   Have them check by multiplication to see whether the factoring is correct.

   $$(3x + 1)(x + 2) = 3x^2 + 7x + 2 \quad \text{It checks.}$$

   Pupils conclude that the correct factors of $3x^2 + 7x + 2$ are $(3x + 1)(x + 2)$.

3. Factor $2x^2 + 9x - 5$

   a. What are the first terms of the binomial factors?

   b. What is the product of the last terms?
   What are the pairs of factors of -5?

   c. Which of the following pairs of factors would you select? Why?

   $(2x + 1)(x - 5)$
   $(2x - 1)(x + 5)$
   $(2x + 5)(x - 1)$
   $(2x - 5)(x + 1)$

B. Suggested Practice

Factor:

1. $2a^2 + 3a + 1$
2. $3c^2 + 8c + 5$
3. $3x^2 - 14x - 5$
4. $2r^2 - r - 3$
5. $2y^2 + y - 15$
6. $2t^2 - 9t + 4$
7. $3x^2 + 11x - 20$
8. The area of a rectangle is $2x^2+5x-12$. Express the length and the width each as a binomial in x.
9. What are the integral values of m that will make the trinomial $3x^2 + mx + 3$ the product of two binomials?

VII. Complete Factoring

A. Suggested Procedure

1. Have pupils consider the expression $4x^2 - 16$. Ask them to factor it. Some may find the greatest common factor. Others may factor it as the difference of two squares. Thus,

   a. $4x^2 - 16 = 4(x^2 - 4)$

   b. $4x^2 - 16 = (2x + 4)(2x - 4)$

   Have pupils observe that the factoring in a is incomplete, since $x^2 - 4$ may be factored further as the difference of squares.

   Have them see that the factoring in b is incomplete, since each of the factors has a common monomial factor.

2. Have pupils consider how $4x^2 - 16 = 4(x^2-4)$ may be factored further.

   $$4x^2 - 16 = 4(x^2-4) = 4(x+2)(x-2)$$

   Have them realize that the factorization is now complete, since each of the binomial factors cannot be factored further.

3. Have pupils factor $2a^2 + 6a + 4$ completely.

$$2a^2 + 6a + 4 = 2(a^2 + 3a + 2) = 2(a + 1)(a + 2)$$

Why is this factorization complete?

Check by multiplication.

4. Guide pupils to the use of the following steps in complete factoring.

   a. Look for a common factor first. Express in factored form. Then examine each factor.

   b. If one of these is a binomial factor, see if it is the difference of two squares. If so, factor it.

   c. If one of these is a trinomial factor, see if it can be factored. If so, factor it.

   d. Make sure the binomial or trinomial factors cannot be factored further.

B. Suggested Practice

Factor each expression completely.

1. $3x^2 - 3y^2$
2. $2b^2 - 8$
3. $2x^2 - 12x + 10$
4. $5a^2 - 20a - 25$
5. $2a^2 + 6a + 8$
6. $a^3 - a$
7. $4x^2 - 24x + 36$
8. $a3 - ab^2$
9. $R^2 - r^2$
10. $x^4 - y^4$

VIII. Using Factoring in Solving Equations

A. Suggested Procedure

1. Have pupils see the need for finding methods of solving equations other than linear equations.

   a. Pose problem: The length of a rectangle is three inches more than its width. The area of the rectangle is 40 square inches. What are its dimensions?

b. Have pupils describe the conditions of the problem by means of an equation, as follows:

Let x represent the number of inches in width
$x+3$ represents the number of inches in length

$x(x+3) = 40$ Area of a rectangle = length x width

Have them observe that this is not a linear equation since it cannot be put in the form $ax + by = c$, (a and b not both 0).

Have them try to solve the equation by methods they have used for solving linear equations. They find they cannot.

c. Have pupils note that in the equation $x(x+3) = 40$, the product of two factors is 40. Have them try to determine a pair of factors of 40 such that one factor is three more than the other.

After trying various pairs of factors of 40, they will eventually find that the required factors are 5 and 8, and they will then be able to solve the problem. They will realize, however, that it is not always easy to find the factors of a number when these factors have conditions imposed upon them.

2. Guide pupils to realize that when the product of two factors is 0, the task of determining the factors is greatly simplified since we now have additional information about one of the factors.

a. Have pupils understand that if the product of two (or more) factors is zero, then at least one of the factors is zero.

1) Review the multiplicative property of zero.

$(6)(0) = 0$ $(0)(0) = 0$

$(-8)(0) = 0$ $(0)(-\frac{6}{5}) = 0$

To generalize:

For every number a, $a \cdot 0 = 0$
$0 \cdot a = 0$

2) Have pupils consider non-zero factors.

$(+2)(+3) = +6$
$(-2)(-3) = +6$
$(-2)(+3) = -6$
$(+2)(-3) = -6$

Have them try other pairs of factors to see whether the product of two non-zero numbers is ever zero. They conclude it is not.

3) Have pupils consider the open sentence $a \cdot b = 0$.

What replacements for a and b from the domain of signed numbers make this true?

Elicit that if $a \cdot b = 0$, then either $a = 0$, or $b = 0$, or $a = 0$ and $b = 0$.

b. Have pupils practice the following:

1) If $a \cdot b = 0$

$b = 6$

Then $a = ?$

2) If $a \cdot b = 0$

$a = 10$

Then $b = ?$

3) If $a \cdot b = 0$

$b = 0$

Then $a = ?$

(a can be replaced by any member of the replacement set)

4) If $a \cdot b = 12$

Then can $a = 0$? (No)
can $b = 0$? (No)

5) What is the value of $2(x - 5)$ when $x = 5$?

6) What is the value of $x(x+3)$ when $x = 0$? When $x = -3$?

7) What is the value of $(x + 4)(x - 3)$ when $x = -4$? When $x = 3$?

8) What value of x will make the first factor of $(x - 2)(x - 1)$ zero? With this value of x, $(x - 2)(x - 1) = ?$

What value of x will make the second factor zero? With this value of x, $(x - 2)(x - 1) = ?$

9) If $x(x+3) = 0$ and $x \neq 0$, what can you say about $x + 3$? about x?

10) If $(x + 5)(x - 4) = 0$, and $x - 4 \neq 0$, what can you say about $x + 5$? about x?

3. Have pupils learn to solve equations involving factorable polynomials.

a. Have pupils understand the meaning of polynomial equations. Consider the following equations:

1) $x + 2 = 9$
2) $y^2 + 5y = 17$
3) $t^3 + 3 = 0$
4) $a^3 = 2a^2 + 5a$

What common property do they have? In each case the expression before and after the equals sign is a polynomial. Such equations are called polynomial equations. Have pupils construct several other polynomial equations.

Have them construct several equations which are not polynomial equations as, for example, $x^2 + \frac{1}{x} = 5$.

b. Guide pupils to an understanding of the degree of a polynomial equation.

1) Review the degree of a polynomial.

2) Which is the polynomial of higher degree in $x + 2 = 9$, $x + 2$ or 9? $(x + 2)$

Since this is of degree one, the equation, $x + 2 = 9$, is called a first-degree equation.

3) Which is the polynomial of higher degree in $y^2 + 5y = 17$, $y^2 + 5y$ or 17? $(y^2 + 5y)$

Since this is of degree two, the equation $y^2 + 5y = 17$ is called a second-degree equation.

4) Of what degree is $t^3 + 3 = 0$? Why?
Of what degree is $a^3 = 2a^2 + 5a$? Why?

c. Have pupils learn how factoring may be used to solve equations with factorable polynomials.

1) Solve $x(x + 3) = 40$. This is the equation which describes the conditions of the problem posed in VIII-A-1-a.

Pupils have previously solved this equation by a trial-and-error procedure. Have them now solve it by using their knowledge of factors whose product is zero. Have them realize that to apply this knowledge, the equation must be written so that one member is zero.

$$\begin{aligned} x(x+3) &= 40 \\ x(x+3)-40 &= 0 \qquad \text{Why is this equivalent?} \\ x^2+3x-40 &= 0 \\ (x+8)(x-5) &= 0 \end{aligned}$$

Then, $x + 8 = 0$ or $x - 5 = 0$
Therefore $x = -8$ or $x = 5$

The solution set of the equation is $\{-8, 5\}$.

Have pupils check each solution in the original equation $x(x+3) = 40$.

Note: In terms of the problem, the domain of x would be restricted to positive numbers. Then -8 must be rejected as a solution of the equation. Accordingly, the solution set of the equation is $\{5\}$. Thus, the width of the rectangle is 5 inches and its length is 8 inches.

2) Solve: $x^2 - 4x = 0$
$x(x-4) = 0$

Then, $x = 0$ or $x - 4 = 0$ Why?

Therefore, $x = 0$ or $x = 4$.

Then the solution set is $\{0, 4\}$.

Check each apparent solution in the original equation.

$$\begin{array}{lcl} 0^2 - 4(0) \overset{?}{=} 0 & \qquad & (4)^2 - 4(4) \overset{?}{=} 0 \\ 0 - 0 \overset{?}{=} 0 & & 16 - 16 \overset{?}{=} 0 \\ 0 = 0 & & 0 = 0 \end{array}$$

Why is $x^2 = 4x$, $x = 4$ not a correct method? (We cannot divide both sides of an equation by a variable if zero is a possible replacement for that variable.)

3) Solve: $2x^2 - 8x + 8 = 2$
$2x^2 - 8x + 6 = 0$ (an equivalent equation with 0 as one member)

Have pupils see that there is a common monomial factor, 2, of the terms of the polynomial $2x^2 - 8x + 6$. Since this is a non-zero number, we may divide both members of the equation by 2. Then we have

$x^2 - 4x + 3 = 0$ or $2(x^2 - 4x + 3) = 0$
$(x-3)(x-1) = 0$

Then $x - 3 = 0$ or $x - 1 = 0$ Since $2 \neq 0$, then $x - 3 = 0$ or $x - 1 = 0$

Therefore, $x = 3$ or $x = 1$ Therefore, $x = 3$ or $x = 1$

The solution set is $\{3, 1\}$.

Have pupils check the solutions in the original equation.

4) Solve: $z^2 = 16$

Have pupils realize that if $z^2 = 16$, then $z = +4$ or -4. These are the solutions. Will the solution by factoring give the same answers?

$$z^2 - 16 = 0$$
$$(z+4)(z-4) = 0$$

Then, $z + 4 = 0$ or $z - 4 = 0$

$z = -4$ $\quad$ $z = 4$

The solution set is $\{-4, 4\}$.

Have pupils check.

5) Solve: $x^2 - 8x + 16 = 0$
$(x-4)(x-4) = 0$

Then, $x - 4 = 0$ or $x - 4 = 0$

$x = 4$ $\quad$ $x = 4$

Have pupils note that since the polynomial on the left side is a trinomial square, the factors are identical and we obtain the same root, 4, twice. It is therefore called a double root, but is written only once as a member of the solution set.

Inform pupils that in courses in higher mathematics reasons will be discovered for considering it to be a double root.

Have pupils check.

6) Discuss with pupils the number of solutions they expect a first-degree equation to have; a second-degree equation to have.

From the above examples, have pupils conclude that a second-degree equation has two roots. (The roots may be the same number.)

B. Suggested Practice

Find the solution set of each of the following polynomial equations.

1. $3(x - 5) = 0$
2. $4x - 20 = 0$
3. $x(x - 3) = 0$
4. $(x + 4)(x - 2) = 0$
5. $r^2 - 3r = 0$
6. $y^2 - 9y + 8 = 0$
7. $a^2 + 4a = 12$
8. $t^2 - 11t = -18$
9. $b^2 - 4 = 0$
10. $4x^2 = 36$
11. $2a^2 = 50$
12. $2y^2 = 24y - 72$
13. $x(x + 3) = 10$
14. $(x - 4)(x + 2) = 16$

OPTIONAL

15. $(x - 2)(x + 5)(x - 10) = 0$
16. $2x^3 - 2x^2 - 4x = 0$

IX. Using Factoring in Problem Solving

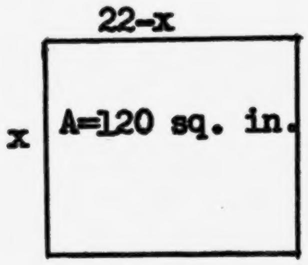
Solve the following:

1. The sum of a certain number and its square is 56. Find the number.

2. The width of a rectangle is 4 inches less than its length. If the area of the rectangle is 117 square inches, what are the dimensions?

3. The perimeter of a rectangle is 44 inches. Its area is 120 square inches. What are the dimensions of the rectangle?

Illustrative Solution:



Let x = number of inches in one dimension

If the perimeter of the rectangle is 44 inches, then half the perimeter or the sum of one length and one width is 22 inches.

Then, $22 - x$ = number of inches in the other dimension

$$x(22 - x) = 120$$
$$22x - x^2 = 120$$
$$x^2 - 22x + 120 = 0$$
$$(x-12)(x-10) = 0$$
$$x - 12 = 0 \quad \text{or} \quad x - 10 = 0$$

Therefore, $x = 12$ or $x = 10$

If $x = 12$, then $22 - x = 10$
If $x = 10$, then $22 - x = 12$

The dimensions of the rectangle are 12 feet by 10 feet.

Check the solution against the conditions of the problem.

4. If one side of a square is increased by 4 inches and an adjacent side is decreased by 4 inches, the area of the resulting rectangle is 20 square inches. What is the length of a side of the square?

5. The altitude of a parallelogram is 3 units less than the base. The area is 54 square units. Find the base and the altitude. (The negative solution has no meaning.)

6. The square of a certain integer is 5 more than the next consecutive integer. Find the number.

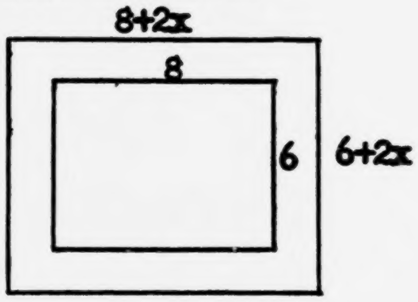
7. A window screen is 10 inches longer than it is wide. Its area is 375 square inches. Find its dimensions.

8. The sum of the squares of two consecutive numbers is 113. Find the numbers.

9. John wishes to double the area of his garden by increasing the length and the width by the same amount. If the original garden is 18 feet long and 12 feet wide, by how many feet must each dimension be increased?

10. A rectangular garden is 6 feet by 8 feet. About this, a rectangular walk of uniform width is built as shown in the diagram.

The area of the walk is 72 square feet. Find the outside dimensions of the walk.



11. The area of a circle is 154 square inches. Find the radius and the diameter. (Use $\frac{22}{7}$ as an approximation for .)

12. Select additional problems from textbooks.

CHAPTER X

FRACTIONS

This section suggests procedures and materials for helping pupils gain understanding and skill in extending fundamental operations to fractional expressions, and in solving equations with fractions.

I. Review and Extension of the Meaning of Fractional Numbers

Note: Although a fraction is a numeral which represents a fractional number, in the future the word fraction will be used to mean the numeral or the number. The context will indicate which meaning is intended.

A. Suggested Procedure

1. Review with pupils the various uses of fractions that they have encountered.

   a. Part of a whole: $\frac{2}{3}$

   b. Ratio of two numbers: 2 to 3, 2:3, $\frac{2}{3}$

   c. Indicated division of two numbers: If 2 is divided by 3, we can write the result in the form of the fraction $\frac{2}{3}$.

2. Have pupils see that when variables are used to represent numbers, we can use fractions in the same ways.

   How do we indicate, by means of a fraction:

   a. The ratio of 2 to x $(\frac{2}{x})$; y to 5 $(\frac{y}{5})$; a to b $(\frac{a}{b})$.

   b. The part of a job that can be done in 1 hour if it takes x hours to do the complete job $(\frac{1}{x})$; the part that can be done in 4 hours.

   c. x divided by 3 $(\frac{x}{3})$

   d. 1 divided by b $(\frac{1}{b})$

   e. a divided by b $(\frac{a}{b})$

   f. $(x^2+5) \div x \quad (\frac{x^2+5}{x})$

3. Have pupils understand that when x represents a signed number, an expression such as $\frac{y^2 + 2y}{y - 2}$ indicates a division with divisor $y - 2$.

4. Have pupils realize that since division by 0 has no meaning, a fraction with a denominator of zero has no meaning.

   For what values of the variable are the above fractions meaningless? Why?

   Note: Every polynomial may be considered a fractional expression with denominator 1.

   Thus, $2x + 5 = \frac{2x + 5}{1}$.

B. Suggested Practice

Express as fractions. Give the value, if any, of the variable for which the fraction has no meaning.

1. x divided by 8
2. The ratio of the length, l, of a rectangle to its width, w.
3. x divided by y
4. $-10 \div a$
5. The part of a job that can be done in x hours if it takes 5 hours to do the complete job.
6. $-10 \div (a - 2)$
7. $(y + 4) \div y^2$
8. $2b \div (b - 5)$
9. The cost of 1 apple if n apples cost 50 cents.
10. $1 \div (x + 1)$
11. $(r + 3) \div (r - 3)$
12. $(a - 2) \div (a2 - 4)$

II. Simplifying Fractions

A. Suggested Procedure

1. Review with pupils that in multiplying the fractions of arithmetic, the numerators are multiplied to give the numerator of the product, and the denominators are multiplied to give the denominators of the product.

$$5 \cdot \frac{3}{4} = \frac{5 \cdot 3}{4} = \frac{15}{4}$$

$$\frac{2}{3} \cdot \frac{1}{5} = \frac{2 \cdot 1}{3 \cdot 5} = \frac{2}{15}$$

$$\frac{3}{4} \cdot \frac{5}{7} = \frac{3 \cdot 5}{4 \cdot 7} = \frac{15}{28}$$

In general,

$\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}$ is a true statement for any replacement for

a, b, c, d by numbers, except $b = 0$ or $d = 0$.

Note: In the future, we will assume the denominator of a fraction is not zero.

2. Have pupils recall that any arithmetic fraction which has the same numerator and denominator is equivalent to the number 1. Thus,

$$\frac{2}{2} = 1 \qquad \frac{4}{4} = 1 \qquad \frac{.3}{.3} = 1$$

In general,

$\frac{x}{x} = 1$ is true for any number replacement of x, except $x = 0$.

3. Have pupils see that in simplifying arithmetic fractions (finding a simpler fraction equivalent to a given one), use is made of the multiplicative identity, the number 1.

For example,

$$\frac{3}{6} = \frac{3 \cdot 1}{3 \cdot 2} = \frac{3}{3} \cdot \frac{1}{2} = 1 \cdot \frac{1}{2} = \frac{1}{2}$$

$$\frac{12}{20} = \frac{2 \cdot 2 \cdot 3}{2 \cdot 2 \cdot 5} = \frac{2}{2} \cdot \frac{2}{2} \cdot \frac{3}{5} = 1 \cdot 1 \cdot \frac{3}{5} = \frac{3}{5}$$

However, in simplifying an arithmetic fraction, it is not always necessary or desirable to factor the numerator and denominator into primes. Thus, noting that 4 is the largest factor common to both numerator and denominator in the fraction

$\frac{12}{20}$, we would factor and simplify as follows:

$$\frac{12}{20} = \frac{4\cdot 3}{4\cdot 5} = \frac{4}{4} \cdot \frac{3}{5} = 1 \cdot \frac{3}{5} = \frac{3}{5}$$

Note: A fraction is in simplest form if there is no common factor in numerator and denominator other than 1.

4. Have pupils understand that the value of an algebraic fraction is considered to be 1, if its numerator and denominator are the same.

   a. Consider the algebraic fraction: $\frac{2x}{2x}$.

      What is its value when x is replaced by 1? by 3? by $6\frac{1}{2}$? by any number except 0?

   b. Consider the fractions: $\frac{4ab^2}{4ab^2}$ $\quad \frac{-3x^3y}{-3x^3y}$ $\quad \frac{a^2 + 3a}{a^2 + 3a}$

      What is the value of each fraction for any replacement of the variable by numbers, except those replacements that make the denominator equal to zero?

   c. What can you conclude about the value for any algebraic fraction which has the same numerator and denominator?

5. Guide pupils to understand the simplification of algebraic fractions as follows:

   a. Simplify: $\frac{x^3}{x^2}$

$$\frac{x^3}{x^2} = \frac{x\cdot x\cdot x}{x\cdot x} \text{ (Meaning of exponents)}$$

$$\frac{x\cdot x\cdot x}{x\cdot x} = \frac{x}{x} \cdot \frac{x}{x} \cdot x \text{ (Meaning of multiplication)}$$

$$\frac{x}{x} \cdot \frac{x}{x} \cdot = 1\cdot 1\cdot x \text{ (Any non-zero number divided by itself is 1)}$$

$$1\cdot 1\cdot x = x \text{ (1 is the multiplicative identity)}$$

Therefore, $\frac{x^3}{x^2}$ is equivalent to x.

Have pupils try several replacements for x (other than zero) to see whether the same number is obtained.

b. Simplify: $\frac{-15y^4}{3y^2}$

$$\frac{-15y^4}{3y^2} = \frac{-5\cdot 3\cdot y\cdot y\cdot y\cdot y}{3\cdot y\cdot y} \qquad \text{Why?}$$

$$= -5\cdot \frac{3}{3}\cdot\frac{y}{y}\cdot\frac{y}{y}\cdot y\cdot y \qquad \text{Why?}$$

$$= -5\cdot 1\cdot 1\cdot 1\cdot y\cdot y \qquad \text{Why?}$$

$$= -5y^2 \qquad \text{Why?}$$

Therefore, $\frac{-15y^4}{3y^2}$ is equivalent to $-5y^2$.

Have pupils verify for $y = 2$.

6. After several similar examples have pupils see how the work of expressing algebraic fractions in simplest form may be shortened by using the largest common factor of numerator and denominator.

a. Simplify: $\frac{x^3}{x^2}$

The largest common factor of numerator and denominator is x^2.

We factor accordingly.

$$\frac{x^3}{x^2} = \frac{x\cdot x^2}{x^2} = x\cdot \frac{x^2}{x^2} = x\cdot 1 = x$$

However,

$$\frac{x^2}{x^3} = \frac{1\cdot x^2}{x\cdot x^2} = \frac{1}{x}\cdot \frac{x^2}{x^2} = \frac{1}{x}\cdot 1 = \frac{1}{x}$$

b. Simplify: $\frac{-15y^4}{3y^2}$

The largest common factor of numerator and denominator is $3y^2$.

Then,

$$\frac{-15y^4}{3y^2} = \frac{-5y^2\cdot 3y^2}{3y^2} = -5y^2\cdot\frac{3y^2}{3y^2} = -5y^2\cdot 1 = -5y^2$$

c. Simplify: $\frac{8(x + 5)}{2(x + 5)^2} = \frac{4 \cdot 2 \cdot (x + 5)}{(x + 5) \cdot 2 \cdot (x + 5)}$

$$= \frac{4}{x+5} \cdot \frac{2(x + 5)}{2(x + 5)}$$

$$= \frac{4}{x+4} \cdot 1$$

$$= \frac{4}{x+5}$$

Why is $\frac{4}{x+5}$ the simplest form of the original fraction?

Have pupils verify that $\frac{8(x+5)}{2(x+5)^2}$ and $\frac{4}{x+5}$ name the same number when x is replaced by 3; by -2.

Why cannot x be replaced by -5?

d. Simplify: $\frac{4a + 4}{a^2 - 1}$

Expressing numerator and denominator in factored form, we have

$$\frac{4a + 4}{a^2 - 1} = \frac{4(a+1)}{(a+1(a-1)}$$

$$= \frac{4}{a-1} \cdot \frac{a+1}{a+1}$$

$$= \frac{4}{a-1} \cdot 1$$

$$= \frac{4}{a-1}$$

Have pupils verify for $a = 4$. Which replacements for the variable must be excluded?

e. Simplify: $\frac{1 - x}{x - 1}$

$$\frac{1 - x}{x - 1} = \frac{-1(x-1)}{(x-1)}$$

$$= -1 \cdot 1, \text{ or } -1$$

Note: After pupils gain experience in simplification of fractions, it will not be necessary for them to record all of the intermediate steps.

B. Suggested Practice

1. Which of the following are in simplest form? Explain.

a. $\frac{4}{6}$

b. $\frac{3}{4}$

c. $\frac{a}{b}$

d. $\frac{xy}{x^2}$

e. $\frac{2r^2s}{3r}$

f. $\frac{3a - 4b}{a}$

g. $\frac{3(x + 4)}{(x + y)^2}$

h. $\frac{2x^2 + y}{x + y}$

2. Express each fraction in simplest form. Tell which replacements for the variables must be excluded.

a. $\frac{15t}{3t^2}$

b. $\frac{40b^2c}{40bc}$

c. $\frac{5rs^2}{35rs}$

d. $\frac{-25p^2r^2}{30p^3r^3}$

e. $\frac{5a + 5b}{a + b}$

f. $\frac{a - 1}{a^2 - 1}$

g. $\frac{m^2 - 25}{3m + 15}$

h. $\frac{x^2 - 4}{x^2 - 4x + 4}$

i. $\frac{c^2 + 6cd + 9d^2}{c^2 - 9d^2}$

j. $\frac{y^2 - 2y - 8}{y^2 - y - 6}$

k. $\frac{x^2 - 4}{2 - x}$

III. Multiplying Fractions

A. Suggested Procedure

1. Have pupils recall the meaning of multiplication:

$\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}$ is a true statement for any replacement for a, b, c, d except $b = 0$ or $d = 0$ (see II-A-1)

For example, $\frac{3}{4} \cdot \frac{7}{8} = \frac{3 \cdot 7}{4 \cdot 8} = \frac{21}{32}$

2. Guide pupils to see that, if we wish to be consistent, multiplication of algebraic fractions should be performed in the same way:

Multiply the numerators to obtain the numerator of the product and multiply the denominators to obtain the denominator of the product.

a. $\frac{x}{y} \cdot \frac{1}{2} = \frac{x \cdot 1}{y \cdot 2} = \frac{x}{2y}$ $(y \neq 0)$

b. $3y \cdot \frac{5y}{7} = \frac{3y \cdot 5y}{7}$ or $3y \cdot \frac{5y}{7} = \frac{3y}{1} \cdot \frac{5y}{7}$ (Every polynomial may be considered an algebraic fraction with denominator 1)

$= \frac{15y^2}{7}$

$= \frac{3y \cdot 5y}{1 \cdot 7}$

$= \frac{15y^2}{7}$

c. $\frac{2}{3} \cdot \frac{x-1}{x+1} = \frac{2(x-1)}{3(x+1)} = \frac{2x-2}{3x+3}$ $(x \neq -1)$

d. $\frac{n+5}{n+6} \cdot \frac{n-3}{n+4} = \frac{(n+5)(n-3)}{(n+6)(n+4)} = \frac{n^2 + 2n - 15}{n^2 + 10n + 24}$ $(n \neq -6, n \neq -4)$

Note: In all operations involving fractions, discuss the limitations on the domains of the variables.

3. Have pupils simplify the product of algebraic fractions whenever possible.

a. $\frac{5x}{2} \cdot \frac{3}{4x} = \frac{5x \cdot 3}{2 \cdot 4x}$ (Meaning of multiplication)

$= \frac{15 \cdot x}{8 \cdot x}$ (Commutative and Associative Properties)

$= \frac{15}{8} \cdot \frac{x}{x}$

$= \frac{15}{8} \cdot 1$, or $\frac{15}{8}$

For which value of the variable is the fraction meaningless?

Check for x = 2

$\frac{5x}{2} \cdot \frac{3}{4x} \stackrel{?}{=} \frac{15}{8}$

$\frac{5 \cdot 2}{2} \cdot \frac{3}{4 \cdot 2} \stackrel{?}{=} \frac{15}{8}$

$\frac{10}{2} \cdot \frac{3}{8} \stackrel{?}{=} \frac{15}{8}$

$\frac{15}{8} = \frac{15}{8}$

b. $\frac{4x}{3y} \cdot \frac{9y}{8x^2} = \frac{36xy}{24x^2y}$

$= \frac{3 \cdot 12xy}{2x \cdot 12xy}$

$= \frac{3}{2x} \cdot \frac{12xy}{12xy}$

$= \frac{3}{2x} \cdot 1, \text{or } \frac{3}{2x}$

For which values of the variables is the fraction meaningless?

Have pupils check for $x = 3$, $y = 2$.

c. $\frac{4x}{3} \cdot \frac{6}{8x^2} = \frac{24x}{24x^2}$

$= \frac{1}{x} \cdot \frac{24x}{24x}$

$= \frac{1}{x} \cdot 1, \text{ or } \frac{1}{x} \quad (x \neq 0)$

d. $\frac{(b+1)}{(b-1)} \cdot \frac{(b-1)}{(b+3)} = \frac{(b+1)\ (b-1)}{(b-1)\ (b+3)}$

$= \frac{(b+1)(b-1)}{(b+3)\ (b-1)}$

$= \frac{(b+1)}{(b+3)} \cdot \frac{(b-1)}{(b-1)}$

$= \frac{b+1}{b+3} \cdot 1, \text{ or } \frac{b+1}{b+3} \quad (b \neq 1, -3)$

4. Have pupils learn to simplify the multiplication of fractions by first factoring each polynomial whenever possible. Have them practice examples such as the following:

a. $\frac{4x+8}{5} \cdot \frac{x}{x+2} = \frac{4(x+2)}{5} \cdot \frac{x}{x+2}$

$= \frac{4(x+2)x}{5(x+2)}$

$= \frac{4x}{5} \cdot \frac{x+2}{x+2}$

$= \frac{4x}{5} \cdot 1, \text{ or } \frac{4x}{5} \quad (x \neq -2)$

Note to teacher: As pupils show increased understanding of multiplication of fractions, it may not be necessary for them to record each step of their work.

b. $\frac{3y+6}{y^2} \cdot \frac{y^2+y}{y+2}$　　c. $\frac{x^2-4}{x+2} \cdot \frac{1}{x-2}$　　d. $\frac{a-4}{a} \cdot \frac{a+1}{a^2-2a-8}$

B. Suggested Practice

Multiply and give each product in simplest form.

1. $\frac{5}{8} \times \frac{3}{5}$
2. $\frac{6}{7} \times \frac{7}{9} \times \frac{1}{4}$
3. $\frac{5x}{b} \cdot \frac{c}{5x}$
4. $\frac{5a^3}{c} \cdot \frac{3c^2}{10a}$ (Check for $a = 2$, $c = 3$)
5. $\frac{mn}{rs} \cdot \frac{s}{m^2n^2}$
6. $(y - 6) \cdot \frac{y + 6}{2(y - 6)}$ (Check for $y = 8$)
7. $\frac{3p - 3q}{10pq} \cdot \frac{20p^2q^2}{p^2 - q^2}$
8. $\frac{ab}{cd} \cdot \frac{xy}{wz} \cdot \frac{1}{x}$
9. $\frac{a^2 - 6a + 9}{a^2 + 5a + 6} \cdot \frac{a^2 - 9}{a - 3}$ (Check for $a = 4$)
10. $\frac{36b^2}{6(b^2 - 4b + 4)} \cdot \frac{3b - 6}{2b}$ (Check for $b = 7$)
11. $\frac{2x^2}{x^2 - 9} \cdot \frac{3 - x}{x}$

IV. Dividing Fractions

A. Suggested Procedure

1. Have pupils recall the meaning of a reciprocal (multiplicative inverse).

   The reciprocal of 2 is $\frac{1}{2}$, for $2 \cdot \frac{1}{2} = 1$.

   The reciprocal of $\frac{3}{4}$ is $\frac{4}{3}$, for $\frac{3}{4} \cdot \frac{4}{3} = 1$.

   The reciprocal of x is $\frac{1}{x}$, for $x \cdot \frac{1}{x} = 1$.

   What is the reciprocal of a^2b? Why?

   What is the reciprocal of $x + 2$? Why?

   What is the reciprocal of $\frac{y^2 + 5}{y - 3}$? Why?

2. Have pupils recall that to divide by an arithmetic fraction, we multiply by its reciprocal.

$$9 \div \frac{1}{3} = 9 \times \frac{3}{1} = 27$$

$$\frac{2}{3} \div \frac{1}{2} = \frac{2}{3} \times \frac{2}{1} = \frac{4}{3}$$

$$\frac{5}{6} \div \frac{3}{4} = \frac{5}{6} \times \frac{4}{3} = \frac{10}{9}, \text{ etc.}$$

In general, $\frac{a}{b} \div \frac{c}{d}$ can be replaced by $\frac{a}{b} \times \frac{d}{c}$. That is, $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$ for any number replacement for a, b, c, d except $b = 0$, $c = 0$, $d = 0$.

3. Guide pupils to see that, if we wish to be consistent, division of algebraic fractions should be performed in the same way:

To divide by an algebraic fraction, multiply by its reciprocal.

a. $\frac{a^2}{d} \div \frac{a}{d} = \frac{a^2}{d} \cdot \frac{d}{a}$ ($\frac{d}{a}$ is the reciprocal of $\frac{a}{d}$)

$$= a \cdot \frac{a}{a} \cdot \frac{d}{d}$$

$$= a \cdot 1 \cdot 1, \text{ or } a$$

b. $\frac{5a+1}{5} \div \frac{5a+1}{6} = \frac{5a+1}{5} \cdot \frac{6}{5a+1}$ ($\frac{6}{5a+1}$ is reciprocal of $\frac{5a+1}{6}$)

c. $\frac{y^2+2y+1}{3y} \div \frac{y^2-1}{6y^2} = \frac{y^2+2y+1}{3y} \cdot \frac{6y^2}{y^2-1}$

Complete and check by letting $y = 2$.

B. Suggested Practice

Divide and check for values of the variables that will make the fractions meaningful.

1. $\frac{15}{11} \div \frac{5}{22}$
2. $\frac{x}{y} \div \frac{x}{y^2}$
3. $\frac{-6b}{8c^2} \div \frac{30b^2}{32c^3}$
4. $\frac{mn^2}{p} \div \frac{2m^2n^2}{3p^2}$
5. $\frac{r-t}{4} \div \frac{r-t}{2}$

6. $\frac{a^2 - b^2}{ab} \div \frac{a - b}{-a}$

7. $\frac{3y + 2}{2y + 3} \div \frac{6y + 4}{6y + 9}$

8. $\frac{x^2 + 2xy + y^2}{x^2 - 2xy + y^2} \div \frac{x + y}{x - y}$

9. $\frac{d^2 + 2d + 1}{5d - 5} \div \frac{d + 1}{15}$

10. $\frac{r^2 + 16r + 63}{r^2 - r - 12} \div \frac{r^2 - 81}{r^2 - 16}$

11. $\frac{y^2 - 4}{3y} \div \frac{2 - y}{6y^2}$ Solution: $\frac{y^2 - 4}{3y} \div \frac{2 - y}{6y^2} = \frac{y^2 - 4}{3y} \cdot \frac{6y^2}{2 - y} = \frac{(y+2)(y-2) \cdot 2y \cdot 3y}{-1(y-2)(3y)}$

$$= \frac{2y(y+2)}{-1} \cdot \frac{y-2}{y-2} \cdot \frac{3y}{3y}$$

$$= -2y(y+2)$$

V. Adding and Subtracting Fractions

A. Suggested Procedure

1. Combining fractions with the same denominator (like fractions).

a. Have pupils review adding (subtracting) arithmetic fractions with same denominators.

$$\frac{5}{4} + \frac{2}{4} = \frac{5 + 2}{4} \text{ or } \frac{7}{4}$$

$$\frac{3}{8} + \frac{2}{8} = \frac{3 + 2}{8} \text{ or } \frac{5}{8}$$

Have them recall that the justification for this is the distributive property. For example,

$$\frac{5}{4} + \frac{2}{4} = 5 \cdot \frac{1}{4} + 2 \cdot \frac{1}{4} = (5+2)\frac{1}{4} = 7 \cdot \frac{1}{4} = \frac{7}{4}$$

Guide pupils to realize that $\frac{a}{b} + \frac{c}{b} = \frac{1}{b} \cdot a + \frac{1}{b} \cdot c = \frac{1}{b}(a+c) = \frac{a+c}{b}$

Therefore, $\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b}$ is a true statement for all replacements of a, b, c by numbers, except $b = 0$.

b. Have pupils see how the distributive property is used as the basis for combining algebraic fractions.

1) $\frac{2}{x} + \frac{3}{x} = 2 \cdot \frac{1}{x} + 3 \cdot \frac{1}{x}$ Why?

$= (2 + 3)\frac{1}{x}$ Why?

$= 5 \cdot \frac{1}{x}$, or $\frac{5}{x}$ Why?

Then, $\frac{2}{x} + \frac{3}{x} = \frac{2 + 3}{x}$ or $\frac{5}{x}$

2) $\frac{3a}{4} - \frac{a}{4} = 3a \cdot \frac{1}{4} - a \cdot \frac{1}{4}$

$= (3a - a)\frac{1}{4}$

$= 2a \cdot \frac{1}{4}$

$= \frac{2a}{4}$, or $\frac{a}{2}$

Then, $\frac{3a}{4} - \frac{a}{4} = \frac{3a - a}{4} = \frac{2a}{4}$, or $\frac{a}{2}$

c. Have pupils practice the following:

1) $\frac{3}{8} + \frac{2}{8} = \frac{3 + 2}{8} = \frac{5}{8}$

2) $\frac{11}{16} - \frac{5}{16} = \frac{11 - 5}{16} = \frac{6}{16} = \frac{3}{8}$

3) $\frac{9}{10} - \frac{3}{10} + \frac{7}{10} = \frac{9 - 3 + 7}{10} = \frac{13}{10}$

4) $\frac{5a}{12} + \frac{7a}{12} = \frac{5a + 7a}{12} = \frac{12a}{12} = a$

5) $\frac{6}{5b} + \frac{1}{5b} - \frac{3}{5b} = \frac{6 + 1 - 3}{5b} = \frac{4}{5b}$ $(b \neq 0)$

6) $\frac{2(x - 2)}{3} - \frac{(x - 2)}{3} = \frac{2(x - 2) - (x - 2)}{3} = \frac{2x - 4 - x + 2}{3} = \frac{x - 2}{3}$

7) $\frac{n}{n-3} + \frac{n}{n-3}$

8) $\frac{5b^2 - b + 7}{5} + \frac{2b^2 - 4}{5}$

9) $\frac{7x^2 - 3x + 5}{8} - \frac{3x^2 - x - 10}{8}$ (To avoid errors in signs, have pupils enclose $3x^2 - x - 10$ within parentheses.)

2. Combining fractions with unlike denominators

a. Review the identity element for multiplication in expressing fractions as equivalent fractions.

1) Change $\frac{3}{4}$ to an equivalent fraction having a denominator of 8.

$$\frac{3}{4} = \frac{3}{4} \cdot 1 \quad \text{(1 is the multiplicative identity)}$$

$$\frac{3}{4} \cdot 1 = \frac{3}{4} \cdot \frac{2}{2} \quad (1 = \frac{2}{2}\text{. Why was } \frac{2}{2} \text{ chosen as a different numeral for 1 rather than } \frac{3}{3} \text{ or } \frac{4}{4}\text{?)}$$

$$\frac{3}{4} \cdot \frac{2}{2} = \frac{3 \cdot 2}{4 \cdot 2}, \text{ or } \frac{6}{8} \quad \text{(Multiplication of fractions)}$$

Then, $\frac{3}{4} = \frac{6}{8}$

Have pupils check by expressing $\frac{6}{8}$ in simplest form to see whether it equals $\frac{3}{4}$.

2) Change $\frac{5a}{3}$ to an equivalent fraction having a denominator of 9.

$$\begin{aligned} \frac{5a}{3} &= \frac{5a}{3} \cdot 1 \\ &= \frac{5a}{3} \cdot \frac{3}{3} \quad \text{(Why was } \tfrac{3}{3} \text{ used to replace 1?)} \\ &= \frac{5a \cdot 3}{3 \cdot 3}, \text{ or } \frac{15a}{9} \end{aligned}$$

Check by expressing $\frac{15a}{9}$ in simplest form.

3) Change $\frac{3x}{4y}$ to an equivalent fraction having a denominator of 20ay.

$$\begin{aligned} \frac{3x}{4y} &= \frac{3x}{4y} \cdot 1 \\ &= \frac{3x}{4y} \cdot \frac{5a}{5a} \quad \text{(Why was } \tfrac{5a}{5a} \text{ used to replace 1?)} \\ &= \frac{15ax}{20ay} \end{aligned}$$

Check by expressing $\frac{15ax}{20ay}$ in simplest form.

4) Practice the following:

$\frac{5}{7} = \frac{?}{21}$ $\qquad$ $\frac{2a}{5} = \frac{?}{25}$ $\qquad$ $\frac{8}{x} = \frac{?}{x^2}$ $\qquad$ $\frac{3by}{xy} = \frac{?}{ax^2y}$

b. Have pupils understand how to combine unlike fractions with numerical denominators, as follows:

1) Find the sum of $\frac{1}{2}$ and $\frac{1}{3}$. Pupils will know from their background in arithmetic, that to add fractions with unlike denominators, we must first find equivalent expressions for each with a common denominator.

Have them recall that the common denominators of two (or more) fractions are the common multiples of the denominators. Thus, in the set of positive integers,

$\{0, 2, 4, 6, 8, 10, 12, 14, 16, \dots\}$ is the set of multiples of 2 (the denominator of the fraction $\frac{1}{2}$). This is so because each element in the set has 2 as a factor.

$\{0, 3, 6, 9, 12, 15, 18, \dots\}$ is the set of multiples of 3 (the denominator of the fraction $\frac{1}{3}$) since each element in this set has 3 as a factor.

The common elements of the two sets $0, 6, 12, 18, \dots$ constitute a set of common multiples of 2 and 3. The elements of this set are the common denominators of the fractions $\frac{1}{2}$ and $\frac{1}{3}$. Whereas any common denominator (any common multiple of the denominators) can be used to express $\frac{1}{2}$ and $\frac{1}{3}$ as like fractions, for convenience the lowest common denominator, namely 6, is used.

Then, $\frac{1}{2} = \frac{1}{2} \cdot \frac{3}{3}$ or $\frac{3}{6}$

$\frac{1}{3} = \frac{1}{3} \cdot \frac{2}{2}$ or $\frac{2}{6}$

$\frac{1}{2} + \frac{1}{3} = \frac{3}{6} + \frac{2}{6} = \frac{3+2}{6}$ or $\frac{5}{6}$

Note: When numbers have no common factors, their least common multiple is their product.

Note: The smallest number in the set of common multiples is zero, since every natural number is a factor of zero. However, zero as a common multiple is not useful in mathematics.

2) Find: $\frac{a}{3} - \frac{a}{4}$

The least common multiple of the denominators 3 and 4 is 12. Then to combine the fractions, we must find equivalent expressions for $\frac{a}{3}$ and $\frac{a}{4}$ with denominator 12. Thus,

$$\frac{a}{3} - \frac{a}{4} = \frac{a}{3} \cdot \frac{r}{4} - \frac{a}{4} \cdot \frac{3}{3}$$

$$= \frac{4a}{12} - \frac{3a}{12}$$

$$= \frac{4a - 3a}{12}, \text{ or } \frac{a}{12}$$

3) Have pupils practice examples such as the following:

Add or subtract as indicated.

$\frac{y}{2} - \frac{y}{4}$ $\qquad$ $\frac{x}{5} + \frac{2x}{3}$ $\qquad$ $\frac{a}{6} + \frac{3a}{8}$ $\qquad$ $\frac{4x^2}{3} - \frac{x^2}{2}$

c. Have pupils learn to find the lowest common denominator by prime factorization.

1) Add the fractions $\frac{11}{40}$ and $\frac{7}{18}$

What is the lowest common denominator? Is the product of 40 and 18 the LCD? This product, 720, is certainly a common denominator, for it is a multiple of each denominator, but it would be convenient if we could find a smaller number that is a multiple of each denominator, that is, having 40 and 18 as factors.

Have pupils find the prime factors of each denominator.

$40 = 2\cdot2\cdot2\cdot5$ $\qquad\qquad$ $18 = 2\cdot3\cdot3$

The sum of the fractions can now be written as:

$$\frac{11}{40} + \frac{7}{18} = \frac{11}{2\cdot2\cdot2\cdot5} + \frac{7}{2\cdot3\cdot3}, \text{or } \frac{11}{2^3\cdot5} + \frac{7}{2\cdot3^2}$$

What are the prime factors of the LCD? (The LCD must contain 2,3 and 5 as factors.) How many times must 2 be used as a factor in the LCD? (Three times) How many times must 3 be used as a factor? (Twice) How many times must 5 be used as a factor? (Once)

Guide pupils to realize that each factor is used in the LCD the greatest number of times it appears in any denominator.

Then the LCD of the fractions $\frac{11}{40}$ and $\frac{3}{18}$ is $2\cdot2\cdot2\cdot3\cdot3\cdot5$, or $2^3\cdot3^2\cdot5$.

Expressing each fraction as an equivalent fraction with a denominator of $2^3\cdot3^2\cdot5$, we have

$$\frac{11}{40} + \frac{7}{18} = \frac{11}{2^3\cdot5}\cdot\frac{3^2}{3^2} + \frac{7}{2\cdot3^2}\cdot\frac{2^2\cdot5}{2^2\cdot5} \quad \text{Why?}$$

$$= \frac{11\cdot3^2}{2^3\cdot3^2\cdot5} + \frac{7\cdot2^2\cdot5}{2^3\cdot3^2\cdot5} \quad \text{Why?}$$

$$= \frac{99}{360} + \frac{140}{360}$$

$$= \frac{99+140}{360}, \text{ or } \frac{239}{360}$$

2) Combine: $\frac{3x}{90} - \frac{x}{36}$

Factoring each denominator into primes, we have:

$90 = 2 \cdot 3 \cdot 3 \cdot 5$

$36 = 2 \cdot 2 \cdot 3 \cdot 3$

Then, $\frac{3x}{90} - \frac{x}{36} = \frac{3x}{2 \cdot 3 \cdot 3 \cdot 5} - \frac{x}{2 \cdot 2 \cdot 3 \cdot 3}$ or $\frac{3x}{2 \cdot 3^2 \cdot 5} - \frac{x}{2^2 \cdot 3^2}$

The LCD is $2 \cdot 2 \cdot 3 \cdot 3 \cdot 5$, or $2^2 \cdot 3^2 \cdot 5$.

Then, $\frac{3x}{90} - \frac{x}{36} = \frac{3x}{2 \cdot 3^2 \cdot 5} \cdot \frac{2}{2} - \frac{x}{2^2 \cdot 3^2} \cdot \frac{5}{5}$

$$= \frac{6x}{2^2 \cdot 3^2 \cdot 5} - \frac{5x}{2^2 \cdot 3^2 \cdot 5}$$

$$= \frac{6x}{180} - \frac{5x}{180}$$

$$= \frac{6x-5x}{180}, \text{ or } \frac{x}{180}$$

d. Have pupils combine fractions with variables in the denominators.

1) Combine: $\frac{3}{2a} + \frac{5}{3a}$ $a \neq 0$

The factors of 2a are 2 and a.
The factors of 3a are 3 and a.

Taking each factor the greatest number of times it appears in any denominator, we have $2 \cdot 3 \cdot a$, or 6a as the LCD.

Then, $\frac{3}{2a} + \frac{5}{3a} = \frac{3}{2a} \cdot \frac{3}{3} + \frac{5}{3a} \cdot \frac{2}{2}$

$$= \frac{9}{6a} + \frac{10}{6a}$$

$$= \frac{9 + 10}{6a}, \text{ or } \frac{19}{6a}$$

2) Combine: $\frac{2}{15y} - \frac{9}{10y^2}$ $y \neq 0$

The factors of 15y are $5 \cdot 3 \cdot y$

The factors of $10y^2$ are $5 \cdot 2 \cdot y \cdot y$

The LCD is $5 \cdot 3 \cdot 2 \cdot y \cdot y$, or $30y^2$

Solution may be completed as above.

3) Combine: $\frac{x + 1}{2a} + \frac{x - 1}{a} \quad a \neq 0$

The LCD is 2a. Why?

Then, $$\frac{x + 1}{2a} + \frac{x - 1}{a} = \frac{x + 1}{2a} + \frac{x - 1}{a} \cdot \frac{2}{2}$$

$$= \frac{x + 1}{2a} + \frac{2(x - 1)}{2a}$$

$$= \frac{x + 1 + 2(x - 1)}{2a}$$

$$= \frac{x + 1 + 2x - 2}{2a}, \text{ or } \frac{3x - 1}{2a}$$

4) Combine: $2 + \frac{3}{x}$

Expressing 2 as the fraction $\frac{2}{1}$, the example can be written in the form

$\frac{2}{1} + \frac{3}{x}$ The LCD is x

$$\frac{2}{1} + \frac{3}{x} = \frac{2}{1} \cdot \frac{x}{x} + \frac{3}{x}$$

$$= \frac{2x}{x} + \frac{3}{x}$$

$$= \frac{2x + 3}{x}$$

5) Combine: $\frac{3}{x+4} + \frac{5}{x^2 - 16}$

$= \frac{3}{x+4} + \frac{5}{(x+4)(x-4)}$ The LCD is $(x+4)(x-4)$

$$= \frac{3(x-4)}{(x+4)(x-4)} + \frac{5}{(x+4)(x-4)}$$

$$= \frac{3x - 12}{(x+4)(x-4)} + \frac{5}{(x+4)(x-4)} = \frac{3x - 12 + 5}{(x+4)(x-4)} = \frac{3x - 7}{x^2 - 16}$$

B. Suggested Practice

1. $\frac{3}{8} - \frac{1}{4} + \frac{2}{3}$

2. $\frac{5x}{y} - \frac{1}{3}$ (Check $x = 2$, $y = 3$)

3. $\frac{2a}{5} + \frac{3a}{7b}$

4. $\frac{5}{c^3} - \frac{5}{c^2} + \frac{6}{c}$ (Check for $c = 2$)

5. $\frac{7}{mn^2} + \frac{3}{m^2n} - \frac{2}{mn}$

6. $\frac{a+2}{6} - \frac{a+3}{8}$

   Solution:

$$\frac{a+2}{6} - \frac{a+3}{8} = \frac{a+2}{2\cdot 3} - \frac{a+3}{2^3} \text{ The LCD is } 2^3\cdot 3$$

$$= \frac{a+2}{2\cdot 3}\cdot\frac{2^2}{2^2} - \frac{a+3}{2^3}\cdot\frac{3}{3}$$

$$= \frac{4(a+2) - 3(a+3)}{24}$$

$$= \frac{4a+8 - 3a - 9}{24}$$

$$= \frac{a-1}{24}$$

7. $\frac{8}{3b} + 5$

8. $\frac{5x-y}{xy^2} - \frac{2x+3y}{x^2y}$

9. $\frac{3b-2}{b^2} + \frac{4b+1}{2b}$ (Check for any value of b except ?)

10. $\frac{5}{6} + \frac{4x-1}{5} - \frac{x+2}{2}$

11. $\frac{3}{2r+8} - \frac{r}{r+4}$

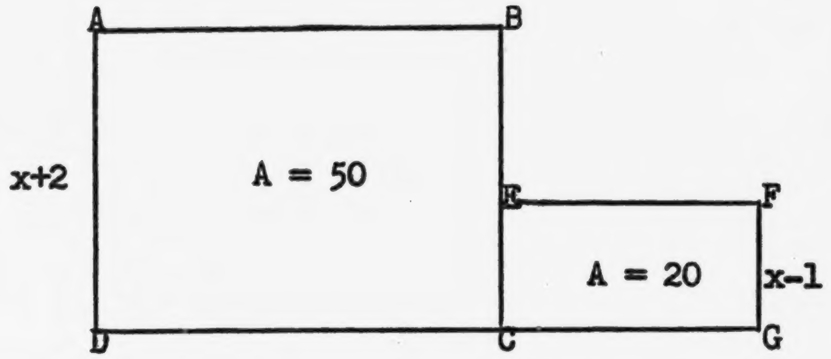
12. See any textbook for additional examples.

13. Express each of the following algebraically:

    a. Mr. Brown left an inheritance of x dollars to be divided among his sons. The first son is to receive $\frac{1}{2}$ of the sum, the second $\frac{1}{3}$, and the third son $\frac{1}{6}$ of the inheritance. Express the sum of the three amounts received by the sons. Will this take care of the entire inheritance?

b. Each of two fractions has a denominator of x. The numerator of the first fraction is 4 more than its denominator, while the numerator of the second fraction is 1 less than its denominator. What is the difference of the first fraction minus the second?

c. Two adjoining vegetable gardens are represented in the diagram below:



If the area of ABCD is 50 and its width is $x + 2$, and the area of EFGC is 20 and its width is $x - 1$, express

1) the length of DC

2) the length of CG

3) the length of DG as a single fraction in simplest form

d. The sides of triangle ABC are $AB = x$, $BC = 2x$, and $CA = 2x + 1$. Another triangle DEF has sides

$DE = \frac{1}{3}$ of AB, $EF = \frac{1}{5}$ of BC, and $FD = \frac{1}{6}$ of CA.

Find the perimeter of triangle DEF in terms of x in its simplest form.

VI. Equations with Fractions

A. Suggested Procedure

1. Review solution of linear equations of type $ax \pm b = c$, $ax \pm b, = cx \pm d$, and $ax \pm bx = c$ (a, b, c, d, integers).

2. Have pupils use the multiplication principle of equations to solve equations with fractions.

   a. Pose problem: Three-fifths of a number exceeds one-tenth of the number by 20. What is the number? Have pupils describe the conditions of the problem by means of an equation as follows:

      Let x represent the number
      Then, $\frac{3}{5}x - \frac{x}{10} = 20$

b. Have pupils realize that it is often desirable, in dealing with equations involving fractions, to transform these equations into equivalent equations having no fractions.

Thus,

$$\frac{3}{5}x - \frac{x}{10} = 20$$

$$10\left(\frac{3x}{5} - \frac{x}{10}\right) = 10(20)$$ Multiplication by 10, the LCD of all fractions in the equation

$$10\left(\frac{3x}{5}\right) - 10\left(\frac{x}{10}\right) = 200$$ Distributive property

$$6x - x = 200$$

$$5x = 200$$

$$x = 40$$

The solution set is $\{40\}$.

Check: It is true that three-fifths of 40 (24) exceeds one-tenth of 40 (4) by 20.

c. Have pupils note that unlike the procedure used in combining fractions, we do not multiply each fraction by 1 when we solve equations. We do not leave each fraction unchanged in value. We change the value of every fraction in the equation and the justification for this is the multiplication principle of equations.

3. Have pupils use the above procedure to solve an equation with fractions when the variable appears in the denominator.

a. Solve: $\frac{2}{3} = \frac{4}{x}$

The LCD of all fractions in the equation is $3x$, if $x \neq 0$.

Then, $(3x)\frac{2}{3} = (3x)\frac{4}{x}$

$$2x = 12$$

$x = 6$ The solution set is $\{6\}$.

Have pupils check: $\frac{2}{3} \stackrel{?}{=} \frac{4}{6}$

$$\frac{2}{3} = \frac{2}{3}$$

b. Solve: $\frac{1}{2y} + \frac{3}{y} = \frac{7}{4}$

The LCD of all fractions in the equation is $4y$, if $y \neq 0$.

$$\begin{aligned} \text{Then, } 4y \cdot \frac{1}{2y} + \frac{3}{y} &= 4y \cdot \frac{7}{4} && \text{Why?} \\ 4y \cdot \frac{1}{2y} + 4y \cdot \frac{3}{y} &= 4y \cdot \frac{7}{4} && \text{Why?} \\ 2 + 12 &= 7y && \text{Why?} \\ 14 &= 7y \\ 2 &= y && \text{The solution set is } \{2\}. \end{aligned}$$

Have pupils check.

c. Solve: $\frac{x+1}{x-2} = \frac{3}{2}$

The LCD of all fractions in the equation is $2(x-2)$, if $x \neq 2$.

$$\begin{aligned} \text{Then, } 2(x-2) \cdot \frac{x+1}{x-2} &= 2(x-2) \cdot \frac{3}{2} \\ 2(x+1) &= (x-2)3 \\ 2x + 2 &= 3x - 6 \\ 8 &= x && \text{The solution set is } \{8\}. \end{aligned}$$

Have pupils check.

Note: When both sides of an equation are multiplied by an expression containing a variable, we do not always obtain an equivalent equation. For example, solve

$$\begin{aligned} \frac{3x-4}{x-2} &= 1 + \frac{2}{x-2} && \text{The LCD is } x-2 \\ (x-2)\left(\frac{3x-4}{x-2}\right) &= (x-2)\left(1 + \frac{2}{x-2}\right) && \text{Multiply both sides by } x-2 \\ 3x-4 &= x-2+2 \\ 2x &= 4 \\ x &= 2 \end{aligned}$$

The solution set of $x=2$ is $\{2\}$. However, 2 is not a solution of the original equation. Hence, $\frac{3x - 2}{x - 2} = 1 + \frac{2}{x-2}$ and the equation obtained by multiplying both sides by x-2 are not equivalent equations. In fact, the original equation has no solution. Its solution set is the empty set.

B. Suggested Practice

1. $\frac{x}{3} + \frac{2x}{5} = 11$
2. $\frac{5a}{4} - \frac{3a}{8} = \frac{7}{2}$
3. $\frac{8b}{9} - \frac{b}{3} + \frac{5b}{6} = 25$
4. $\frac{5}{a} + \frac{3}{2a} = \frac{13}{20}$
5. $\frac{x-5}{10} + \frac{2x-5}{5} = \frac{7}{2}$
6. $\frac{y+8}{6} - \frac{y+2}{3} = -\frac{5}{6}$
7. $5 = \frac{25}{x}$
8. $\frac{3}{x} + 2 = \frac{11}{2x}$
9. $\frac{2w-5}{5} + \frac{w+6}{3} - \frac{4w}{5} = 0$
10. $\frac{y+3}{y-4} = \frac{9}{2}$

11. The denominator of the second of two fractions is 3 times that of the first. The numerator of the first fraction is 3 more than its denominator, while the numerator of the second fraction is 1 less than its denominator. When the second fraction is subtracted from the first, the result is $\frac{5}{3}$. What are the fractions?

12. Harry can wash his father's car in 60 minutes. His older brother can do it in 30 minutes. How long would it take the brothers working together to wash the car?

Illustrative Solution

Let x represent the number of minutes it will take them to wash the car, working together.

$\frac{1}{60}$ is the part of the job Harry can do in 1 minute

$\frac{1}{60} \cdot x$ or $\frac{x}{60}$ is the part Harry can do in x minutes

$\frac{1}{30}$ is the part his brother can do in 1 minute

$\frac{1}{30} \cdot x$ or $\frac{x}{30}$ is the part his brother can do in x minutes

1 = one whole job done by both in x minutes

Then, $\frac{x}{60} + \frac{x}{30} = 1$ The LCD is 60

$$60\left(\frac{x}{60} + \frac{x}{30}\right) = 60 \cdot 1$$

$$x + 2x = 60$$

$$3x = 60$$

$$x = 20$$

It requires 20 minutes for the brothers to wash the car, working together.

Check solution against conditions of the problem.

13. Jean and Evelyn take turns addressing envelopes for their father's business. If it takes Jean 3 hours to do a certain mailing, and it takes Evelyn twice as long to do the same mailing, how long would it take both girls working together to do the job?

14. Mr. Wilson can plow a field in 8 hours, while his hired hand can plow the same field in 10 hours. How long would it take if both worked together? (Assumption: Each uses his own plow.)

15. A set of twin boys working together can paint their room in two hours. If it takes one of them 5 hours working alone, how long should it take the other if he were to do it alone?

16. If it takes 5 minutes to fill a certain bathtub and 9 minutes to empty it, how long will it take to fill the tub if the inlet pipe and the outlet pipe are both open?

17. Alice's father drives to the railroad station, a distance of 15 miles. One day he drove the first 10 miles at a certain speed and the rest of the distance at twice the speed. If the whole trip took 25 minutes, at what speeds was he driving?

18. The distance from Mr. Brown's home to his office is 15 miles. If Mr. Brown drove the first third of the distance at 50 mph and the whole trip took 21 minutes, at what rate did he travel during the rest of the trip?

19. How much should a coat be marked (selling price) if the cost is $91, the overhead is 20% of the selling price, and the profit is 10% of the selling price?

20. The numerator of a fraction is 2 less than the denominator. If 4 is added to the denominator, the result is $\frac{1}{3}$. What is the fraction?

CHAPTER XI

THE REAL NUMBERS

This chapter presents suggested procedures for helping pupils develop some basic concepts of rational and irrational numbers, which, together, form the real number system. Here, too, are suggestions for developing understanding and skill in determining decimal approximations of square roots, and in simplifying radicals.

I. Rational Numbers

A. Suggested Procedure

1. Help pupils understand the meaning of rational number.

   a. Elicit that the numbers pupils have worked with so far include the positive and negative integers and fractions, and zero.

   Have pupils consider some examples of these numbers:

   $\frac{3}{4}$, 2, -8, $4\frac{2}{5}$, 1.7

   Have pupils observe that each of the above numbers can be expressed in the form $\frac{a}{b}$ where a and b are integers and $b \neq 0$.

   $\frac{3}{4}$ is expressed in the form $\frac{a}{b}$, with $a = 3$, $b = 4$.

   2 can be expressed as $\frac{2}{1}$

   -8 can be expressed as $\frac{-8}{1}$

   $4\frac{2}{5}$ can be expressed as $\frac{22}{5}$

   1.7 can be expressed as $\frac{17}{10}$

   b. Inform pupils that numbers which can be expressed in the form $\frac{a}{b}$ where a and b are integers, and $b \neq 0$ are called rational numbers. Have pupils realize that the set of rational numbers is the set of positive and negative integers and fractions, and zero.

   c. Have pupils realize that a rational number has many names. For example,

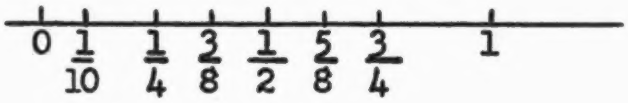
   $$2 = \frac{2}{1} = \frac{4}{2} = \frac{8}{4} \text{ etc.}$$

2. Help pupils visualize the rational numbers as points on a line.

   a. Have them recall that the integers were associated with some points on the number line.

   b. Have pupils realize that certain points on the number line are associated with rational numbers.

      1) Have them see that between any two points associated with two consecutive integers, say 0 and 1, there are many more points on the number line. Have pupils mark some points corresponding to rational numbers between 0 and 1 as in the following diagram:



      2) Have pupils repeat this procedure with other intervals on the number line, say, the interval between 4 and 5, and the interval between -2 and -3.

   c. Have them note that of two rational numbers, the larger number is associated with a point which is to the right of the point corresponding to the smaller number. For example,

$$2\frac{3}{4} > 2\frac{1}{2}$$

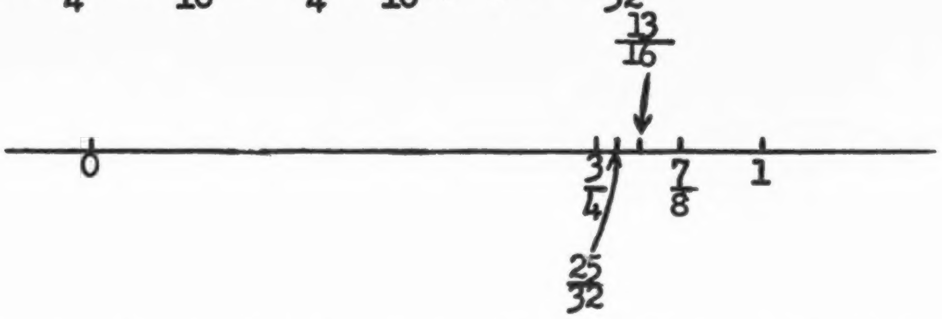
   and the point associated with $2\frac{3}{4}$ is to the right of the point associated with $2\frac{1}{2}$.

3. Have pupils discover that between any two rational numbers there is another rational number. One such number can be found by taking the average of the two numbers.

   Between $\frac{3}{4}$ and 1 is $(\frac{3}{4} + 1) \div 2$, or $\frac{7}{8}$

   Between $\frac{3}{4}$ and $\frac{7}{8}$ is $(\frac{3}{4} + \frac{7}{8}) \div 2$, or $\frac{13}{16}$

   Between $\frac{3}{4}$ and $\frac{13}{16}$ is $(\frac{3}{4} + \frac{13}{16}) \div 2$, or $\frac{25}{32}$



   As the result of the above, pupils should realize that between any two rational numbers there exists an infinite number of rational numbers. Tell pupils that when a set of numbers has this property, the set is said to be dense. Therefore, the set of rational numbers is dense. If we associate every rational number with a point on the number line, have we accounted for every point on the line? It will be shown later that some of the points have not been accounted for.

Note: The set of integers cannot be said to be dense. There is no integer, for example, between 3 and 4.

4. Expressing rational numbers as decimals

   a. Review terminating decimals.

   $\frac{1}{2} = .5$ $\qquad\qquad$ $\frac{3}{4} = .75$ $\qquad\qquad$ $\frac{1}{8} = .125$

   b. Review repeating decimals.

   $$\begin{array}{r} \underline{.333\ldots} \\ 3)1.000 \end{array} \qquad\qquad \begin{array}{r} \underline{.2727\ldots} \\ 11)3.0000 \end{array} \qquad\qquad \begin{array}{r} \underline{.31818\ldots} \\ 22)7.0000 \end{array}$$

   Elicit that the above decimals are non-terminating decimals and also repeating decimals ... the same digit or group of digits repeats unendingly. The repetend is sometimes indicated by a bar above it, e.g.,

   $.\overline{3}$, $\quad .\overline{27}$, and $\ .3\overline{18}$

   c. Have pupils generalize that any rational number can be expressed in decimal form. The decimal will be either a terminating decimal or a repeating decimal. This may be illustrated for the rational number $\frac{1}{7}$, as follows:

   Suppose we are computing the decimal for $\frac{1}{7}$. We begin dividing as follows:

   $$\begin{array}{r} \underline{0.14} \\ 7)1.0000\ldots \\ \underline{7} \\ 30 \\ \underline{28} \\ 20 \end{array}$$

   Note: $\frac{1}{7} = .142857\ldots$

   The first remainder is 3; the second is 2. If ever we again get the remainder 3 or some remainder which has occurred before, the decimal will begin repeating at that point. Since the remainder after each successive division has to be less than the divisor, there are only 7 different remainders possible (0, 1, 2, 3, 4, 5, 6). Then if we continue the division process long enough, we will eventually repeat a remainder.

If the remainder 0 occurs in a division, the division terminates.

5. Expressing decimals as rational numbers

a. Have pupils express several terminating decimals as rational numbers, that is, in the form $\frac{a}{b}$, where a and b are integers, and $b \neq 0$.

$$.25 = \frac{25}{100}$$
$$.706 = \frac{706}{1000}$$
$$1.98 = \frac{198}{100}$$

Pupils will conclude that every terminating decimal represents some rational number.

b. Does every repeating decimal represent a rational number? If so, how do we determine which rational number it represents?

1) $.666 =$? rational number

$$\begin{aligned} \text{Let } N &= .666\ldots \\ 10\,N &= 6.666\ldots \\ N &= .666\ldots \\ 9\,N &= 6 \\ N &= \frac{2}{3} \\ .666\ldots &= \frac{2}{3} \end{aligned}$$

Compare the terminating decimal .666, which is equal in value to $\frac{666}{1000}$, with the repeating decimal $.66\overline{6}$ which is equal in value to $\frac{2}{3}$. Which is larger? Explain.

2) $.\overline{27} =$? rational number

$$\begin{aligned} \text{Let } N &= .\overline{27} \\ 100\,N &= 27.27\ldots \\ N &= .27\ldots \\ 99\,N &= 27 \\ N &= \frac{27}{99} \\ N &= \frac{3}{11} \\ .\overline{27} &= \frac{3}{11} \end{aligned}$$

3) $.1\overline{6} \quad = \quad ?$ rational number

$$\begin{aligned} \text{Let } N &= .166\ldots \\ 10\,N &= 1.666\ldots \\ N &= .166\ldots \\ 9\,N &= 1.5 \\ N &= \frac{1.5}{9} = \frac{15}{90} = \frac{1}{6} \\ N &= \frac{1}{6} \end{aligned}$$

$$.1\overline{6} = \frac{1}{6}$$

or

$$\begin{aligned} \text{Let } N &= .1666\ldots \\ 100\,N &= 16.666\ldots \\ 10\,N &= 1.666\ldots \\ 90\,N &= 15 \\ N &= \frac{15}{90} \text{ or } \frac{1}{6} \\ \text{Then } .1\overline{6} &= \frac{1}{6} \end{aligned}$$

Have pupils conclude that it appears that a repeating decimal can be expressed in the form $\frac{a}{b}$ where a and b are integers $(b \neq 0)$. Therefore, they represent rational numbers. Tell them that this statement is proved in more advanced courses in mathematics.

B. Suggested Practice

1. Give two rational numbers between

   a. 5 and 6   Solution: $5\frac{1}{4}$, 5.85

   b. 7 and 8

   c. -3 and -4

   d. $2\frac{1}{4}$ and $2\frac{7}{8}$

   e. -1 and -1.2

2. Which number in each pair is the greater? (refer to number line)

   a. $\frac{-3}{5}, \frac{-2}{3}$

   b. $\frac{5}{2}, \frac{3}{2}$

   c. $\frac{-2}{4}, \frac{-3}{4}$

   d. $-8, \frac{-33}{4}$

   e. $\frac{-1}{2}, -2$

   f. -2.4 and $-2\frac{1}{3}$

   g. .05 and $\frac{1}{5}$

   h. -1.8, + 1.8

3. Express each of these rational numbers as a repeating or terminating decimal.

   $\frac{7}{8} \qquad \frac{9}{5} \qquad -5\frac{1}{4} \qquad 1\frac{1}{3} \qquad \frac{5}{16}$

4. (OPTIONAL) Express each of the following repeating decimals as a quotient of integers.

   a. .1111...
   b. .0909...
   c. $.\overline{083}$
   d. $.\overline{12}$
   e. 3.75252...

II. Irrational Numbers

A. Suggested Procedure

1. Have pupils construct a non-terminating, non-repeating decimal.

   a. Have pupils recall that a rational number can be expressed as a terminating or repeating decimal, and a repeating or terminating decimal represents a rational number.

      Pose question: Are there any decimals that do not terminate or repeat?

   b. Have pupils examine the following non-terminating decimal numeral:

      .02002000200002...

      They see that it is non-repeating since each succeeding 2 is preceded by an extra zero. Why does this represent a non-rational number?

2. Have pupils make up several non-terminating, non-repeating decimals.

3. Have them learn a new name for non-terminating, non-repeating decimals.

   a. Guide pupils to realize that since non-terminating, non-repeating decimals cannot represent rational numbers, a new name is needed for the numbers they do represent.

   b. Inform pupils that the numbers represented by non-terminating, non-repeating decimals are called irrational numbers. Irrational numbers cannot be expressed in the form $\frac{a}{b}$, where a and b are integers.

4. Have pupils realize that irrational numbers may be positive or negative. For example, .020020002... represents a positive irrational number and -.020020002... represents a negative irrational number.

5. Tell pupils that the set of all rational and irrational numbers is called the set of real numbers.

B. Suggested Practice

1. Tell whether each of the following is rational or irrational.

   a. $.\overline{32}$

   b. .813813381333... (Irrational because it is non-terminating and non-repeating.)

   c. 15.73333...

   d. -2.75

2. Tell why each of the following is a real number:

   a. 1

   b. -2.15

   c. $\frac{469}{873}$

   d. 1.414114111...

III. The Real Numbers

A. Suggested Procedure

1. Have pupils understand how irrational numbers may be approximated by rational numbers.

   a. Have them consider the non-terminating decimal:

   1.242244222444222244444...

   It is non-repeating and therefore represents an irrational number.

   Have pupils see that 1.2 is a 1-decimal-place numeral, the value of which approximates that of the irrational number.

   1.24 is a two-decimal-place rational approximation

   1.242 is a three-decimal-place rational approximation, etc.

   What is a ten-decimal-place rational approximation of the irrational number?

   Which is a closer approximation, a one-decimal-place rational approximation of an irrational number, or a ten-decimal-place approximation of that same number? Why?

2. Have pupils visualize the irrational numbers as points on a number line.

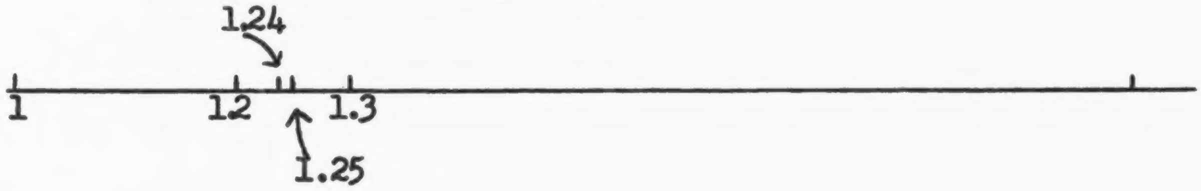
   a. Have them recall that rational numbers may be associated with points on a number line. Are there any points on the number line that can be associated with irrational numbers?

b. Have pupils consider the irrational number represented by the decimal 1.2422442224442222444...

What point on the number line can we associate with this irrational number?

Have them see that the point corresponding to it on the number line lies between:

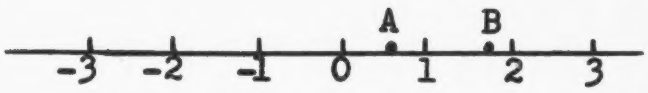
1 and 2
1.2 and 1.3
1.24 and 1.25
1.242 and 1.243
1.2422 and 1.2423 etc.



c. Have pupils note that the segment of the line between the points associated with each successive pair of numbers becomes smaller and smaller. Also, each segment is included or "nested" within the preceding one. As the points associated with each pair of numbers "move" closer together, it appears they will eventually "close in" on one, and only one, point on the line. This point is associated with the irrational number 1.2422442224442222444...

d. Have pupils realize that each real number is associated with a point on the number line, and each point on the number line is associated with a real number (rational or irrational).

e. Have them understand that if two real numbers are associated with points A and B on the number line, the number associated with B is greater than the number associated with A, if B is to the right of A.



3. Tell pupils that just as we add, subtract, multiply and divide with rational numbers, so also can we perform these operations with real numbers.

We assume the properties of these operations which hold for rational numbers also hold for real numbers.

B. Suggested Practice

1. Graph the following pairs of numbers. Then use one of the symbols $<$, $>$, $=$ to write a correct statement comparing the numbers in each pair.

   a. 2.42000... and 1.5333... (2,42000... $>$ 1.5333...)

   b. 1 and -3.555...

   c. 0 and 4.2525

   d. -1.373373337... and 2

2. Arrange this set of real numbers in order from smallest to largest.

   $\left\{ \frac{3}{8}, \quad .3\overline{8}, \quad \frac{3}{8}, \quad .383883888... \right\}$

IV. Meaning of Square Root

A. Suggested Procedure

1. Have pupils review meaning of exponent, base, power, factor, and square.

2. Tell pupils that the square of a rational number is called a perfect square, e.g.,

   $9^2 = 81$, $(-1)^2 = 1$, $(\frac{2}{3})^2 = \frac{4}{9}$ $(-.5)^2 = .25$

   Thus, 81, 1, $\frac{4}{9}$, and .25 are each perfect squares.

   What is the square of 0? ($0^2 = 0$)

   Notice that the perfect squares, except zero, are positive.

3. Introduce concept of square root

   a. Pose problem: The area of a square is 100 sq. ft. What is the length of the side of the square?

      Solution: $A = s^2$ (The domain of s is the set of non-negative rational numbers.)

      $100 = s^2$ Is there a number whose square is 100?

      Since the number which used as a factor twice will give 100 is 10, $s = 10$.

   b. Tell pupils that one of the two equal factors of a number is called a square root of the number. 10 is a square root of 100 because $10 \cdot 10 = 100$.

4. Number of square roots of a number

a. Elicit from pupils that the square of a positive rational number, or of a negative rational number, is a positive number.

$9^2 = 81 \qquad (\frac{3}{4})^2 = \frac{9}{16} \qquad (-\frac{1}{2})^2 = \frac{1}{4} \qquad (-2)^2 = 4$

b. Have pupils realize that since every perfect square, except zero, is the product of two equal positive numbers or of two equal negative numbers, every positive perfect square has two square roots - a positive square root and a negative square root.

The square roots of 49 are +7 and -7 since $(+7)(+7) = 49$ and $(-7)(-7) = 49$. (Zero has only one square root: 0)

How are the two square roots of a positive perfect square related to each other? (They are additive inverses.)

c. The positive square root of a number is called the principal square root. Although +7 and -7 are both square roots of 49, 7 is the principal square root.

5. Have pupils learn the meaning of the symbol, "$\sqrt{}$".

a. Since positive numbers may have two square roots, one positive and one negative, different symbols are used to represent them.

One square root of 16 is +4, and may be represented by $\sqrt{16}$.

The other square root of 16 is -4, and may be indicated by $-\sqrt{16}$.

Then, $(\sqrt{16})^2 = (\sqrt{16})(\sqrt{16}) = 16$,

and $(-\sqrt{16})^2 = (-\sqrt{16})(-\sqrt{16}) = 16$

b. Tell pupils that the symbol $\sqrt{}$ is called a radical. The numeral under the radical sign is called a radicand.

Thus, in the expression $\sqrt{169}$, 169 is the radicand.

c. Have pupils note that although positive numbers may have two square roots, one positive and the other negative, the radical sign with no sign before it represents only the non-negative root. Thus,

$\sqrt{36} = 6$ and $-\sqrt{36} = -6$, or $\pm\sqrt{36} = \pm 6$

$\sqrt{\frac{4}{9}} = \frac{2}{3}$ and $-\sqrt{\frac{4}{9}} = -\frac{2}{3}$, or $\pm\sqrt{\frac{4}{9}} = \pm\frac{2}{3}$

Note: We cannot tell whether the principal square root of a^2 is +a or -a, unless we know whether a is a positive or a negative number.

If $a = 3$, then $\sqrt{a^2} = \sqrt{3^2} = \sqrt{9} = 3 = a$

If $a = -3$, then $\sqrt{a^2} = \sqrt{(-3)^2} = \sqrt{9} = 3 = -a$

Thus, $\sqrt{a^2} = a$ if a is positive or zero.

$\sqrt{a^2} = -a$ if a is negative.

6. Have pupil realize:

   a. If $x^2 = a$, then $x = \pm\sqrt{a}$, where a is a positive number.

   b. $(\sqrt{a})^2 = (\sqrt{a})(\sqrt{a}) = a$; $(-\sqrt{a})^2 = (-\sqrt{a})(-\sqrt{a}) = a$

7. Have pupils consider whether negative numbers have square roots in the set of real numbers.

   a. Does -9, for example, have a square root? That is to say, can we find two equal factors whose product is -9?

      Since $3 \times 3 = 9$ and $(-3) \times (-3) = 9$, neither 3 nor -3 is a square root of -9.

   b. Have pupils recall that the square of a positive number is positive, and the square of a negative number is positive. Also, the square of zero is 0. They conclude that no negative real number has a square root in the set of real numbers. Then -9 has no square root in the set of real numbers.

OPTIONAL

8. The index of a radical

   a. Have pupils consider that some rational numbers may be expressed as the product of three equal factors. For example,

      $64 = 4 \cdot 4 \cdot 4$ or 4^3

      Tell pupils that in such a case, each factor is a cube root of the number, and the number is the cube of the factor. Thus, 64 is the cube, or third power, of 4, and 4 is a cube root of 64. Is -4 also a cube root of 64?

      What is the cube root of 8? of 27? of -27?

      Have pupils realize that -27 does have a cube root in the set of real numbers.

b. Have pupils see how a radical sign may be used to indicate the cube root of a number. The cube root of 64 is written as

$$\sqrt[3]{64}$$

It follows that $\sqrt[3]{64}$ is a number such that

$$\sqrt[3]{64} \cdot \sqrt[3]{64} \cdot \sqrt[3]{64} = 64$$

The small symbol 3 in $\sqrt[3]{64}$ is called the index and indicates that one of three equal factors whose product is 64 is the root we wish to find.

c. The index of $\sqrt[4]{81}$ is 4 and indicates that one of four equal factors whose product is 81 is the root we wish to find.

Since $3 \cdot 3 \cdot 3 \cdot 3 = 81$, 3 is a fourth root of 81. Is -3 a fourth root of 81?

d. Tell pupils that it is understood that the absence of an index means the square root.

$\sqrt{81} = 9$ This could be written $\sqrt[2]{81}$.

The number 2 is the index of $\sqrt{81}$, even though "2" is not usually written with the symbol.

B. Suggested Practice

1. What is the value of each of the following?

a. $\sqrt{1}$ Solution: The principal square root is 1.

b. $\sqrt{16}$

c. $\sqrt{400}$

d. $\sqrt{.01}$

e. $\sqrt{2.25}$

f. $\sqrt{(8)^2}$

g. $\sqrt{(\frac{2}{3})^2}$

h. $\sqrt{(-7)^2}$

i. $\sqrt{(-9)^2}$

2. What does $(\sqrt{4})^2$ equal? $(\sqrt{25})^2$? $(\sqrt{58})^2$?

OPTIONAL

3. State the index and the radicand. Which rational number is represented?

   a. $\sqrt{36}$ Solution: index 2, radicand 36. The integer 6 is represented by $\sqrt{36}$.

   b. $\sqrt[3]{27}$

   c. $\sqrt[3]{1}$

   d. $\sqrt{16}$

   e. $\sqrt[4]{1}$

   f. $\sqrt{\frac{16}{25}}$

4. What does $(\sqrt[3]{10})^3$ equal? $(\sqrt[4]{35})^4$?

V. Approximation of Square Roots

A. Suggested Procedure

1. Review the meaning of the closure property of a set of numbers under an operation. Have pupils recall that the set of rational numbers is closed with respect to the four fundamental operations. Are there any operations for which it is not closed?

2. Have pupils realize certain square roots are irrational.

   a. Pose problem: The area of a square is 2 square feet. What is the length of each side of the square?

   Solution: $A = s^2$ (The domain of s is the set of positive, rational numbers.)

   $$2 = s^2 \text{ or } s^2 = 2, \text{ and } s = \sqrt{2}$$

   What are the replacements for s which make $s^2 = 2$ true?

   The pupils will suspect after a number of trials that no rational number will make $s^2 = 2$ true. It can be proved that this is so.

   Thus, the set of rational numbers is not closed under the operation of extracting square root.

b. Have pupils try to find a rational approximation for $\sqrt{2}$.

1) $1 \times 1 = 1$ Then $\sqrt{2}$ is greater than 1.

   $2 \times 2 = 4$ Then $\sqrt{2}$ is less than 2.

   $1 < \sqrt{2} < 2$ Then either 1 or 2 is a rational approximation of $\sqrt{2}$.

   Can we find two numbers that are closer together than 1 and 2 and still have $\sqrt{2}$ between them?

2) Have pupils try squaring some rational numbers between 1 and 2.

   $1.3 \times 1.3 = 1.69$
   $1.4 \times 1.4 = 1.96$
   $1.5 \times 1.5 = 2.25$

   Have pupils note that $\sqrt{2}$ is greater than 1.4, but less than 1.5.

   $1.4 < \sqrt{2} < 1.5$ Then either 1.4 or 1.5 is a one-decimal place rational approximation of $\sqrt{2}$.

   Can we find a better approximation?

3) $(1.41)^2 = 1.9881$

   $(1.42)^2 = 2.0164$

   Have pupils note that $1.41 < \sqrt{2} < 1.42$.

   Then either 1.41 or 1.42 is a two-decimal place rational approximation of $\sqrt{2}$.

4) This method may be continued to three and four (and more) decimal places for successively closer rational approximations of $\sqrt{2}$.

   $1.414 < \sqrt{2} < 1.415$
   $1.4142 < \sqrt{2} < 1.4143$ etc.

5) Inform pupils that no matter how far this work is carried, the number $\sqrt{2}$ cannot be expressed as a terminating or repeating decimal.

6) Have them conclude that $\sqrt{2}$ is an irrational number. We can find only rational approximations of $\sqrt{2}$.

7) It can be shown that if the square root of an integer is between two consecutive integers, the root is irrational.

For example, $\sqrt{3}$, $\sqrt{5}$, $\sqrt{7}$, $\sqrt{8}$, $\sqrt{10}$, etc. are irrational numbers.

8) Have pupils determine which of the following are rational numbers and which are irrational:

a) $\sqrt{15}$

b) $\sqrt{64}$

c) $\sqrt{30}$

d) $\sqrt{100}$

e) $\sqrt{99}$

f) $\sqrt{49}$

g) $\sqrt{75}$

h) $\sqrt{225}$

i) $\sqrt{39}$

9) Tell pupils that the number π is also an irrational number.

The rational numbers 3.14 and $\frac{22}{7}$ which they have used are only approximations of π.

3. Have pupils approximate square roots by successive division. (Newton's Method)

a. Elicit that the square root of a number is one of two equal factors whose product is the given number. Have them realize that if the two factors of a product are not equal, the square root is somewhere between them.

$6 \cdot 6 = 36$ $\sqrt{36} = 6$

$4 \cdot 9 = 36$ $\sqrt{36}$ is between 4 and 9

$$4 < \sqrt{36} < 9$$

b. Find a rational approximation for $\sqrt{11}$.

1) Estimate:

Since 11 is between the two perfect squares 9 and 16, $\sqrt{11}$ is between $\sqrt{9}$ and $\sqrt{16}$, that is, between 3 and 4.

$$3 < \sqrt{11} < 4$$

2) Divide:

Use 3, the integer whose square is nearest 11 as a guess at $\sqrt{11}$ and divide 11 by 3, carrying the work to one more digit than there are digits of agreement in divisor and quotient.

$$3\overset{3.6}{\overline{)11.0}}$$ We do not round the quotient.

3) Average:

Since the quotient 3.6 is greater than the 3, the guess "3" was too small.

$\sqrt{11}$ is between 3 and 3.6.

$3 < \sqrt{11} < 3.6$

We can obtain a better approximation by taking the average of the original guess and the quotient.

$$\frac{3 + 3.6}{2} = \frac{6.6}{2} = 3.3$$

Now the estimate is 3.3.

4) To find a closer approximation to $\sqrt{11}$, divide 11 by 3.3 and carry the work to one more digit than there are digits of agreement.

$$\begin{array}{r} 3.33 \\ 3.3\overline{)11.000} \\ \underline{9\,9} \\ 1\,10 \\ \underline{99} \\ 110 \\ \underline{99} \\ 110 \end{array}$$

We do not round the quotient.

Then $3.3 < \sqrt{11} < 3.33$

$$\frac{3.3 + 3.33}{2} = \frac{6.63}{2} = 3.31$$

Now the estimate is 3.31. Check: $3.31 \times 3.31 = 10.9561$ (very close to 11)

5) These steps may be repeated to obtain as close an approximation as desired. The approximation is accurate to as many places as match in the divisor and quotient.

Then $\sqrt{11}$ is approximately 3.3 to the nearest tenth. That is,

$\sqrt{11} \approx 3.3$

6) Have pupils approximate to the nearest tenth:

$\sqrt{7}$ $\qquad$ $\sqrt{13}$ $\qquad$ $\sqrt{8}$ $\qquad$ $\sqrt{3.5}$ $\qquad$ $\sqrt{120}$

4. Have pupils learn to use a table of squares and square roots.

a. Inform pupils that mathematicians, scientists, engineers, and others use tables to find squares and square roots. Since the needed values can be read from the tables, a great deal of time is saved.

b. Have pupils use a table to find the square of 26. Locate 26 in the column headed N (Number) and move to the right until the column headed N^2 (Square) is reached. There we read 676. Then $26^2 = 676$.

c. Have pupils use a table to find $\sqrt{26}$. Locate 26 as before. Then move to the right until the column headed $\sqrt{N}$ (Square Root) is reached. There we read 5.099. Then $\sqrt{26} = 5.099$ to the nearest thousandth.

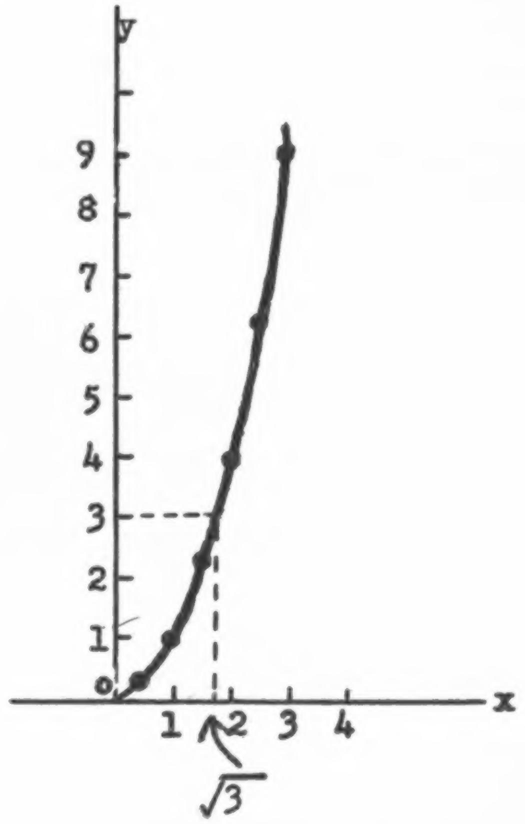
d. Have pupils compare the results obtained when finding square root by successive division with the values listed in the table.

5. (OPTIONAL) Have pupils read square root from a graph.

a. Have them use squared paper (ten to the inch) to graph the set of points (x,y) such that $y = x^2$, where $x \geq 0$.

1) Have pupils plot the points $(0,0)$, $(\frac{1}{2},\frac{1}{4})$, $(1,1)$, $(\frac{3}{2},\frac{9}{4})$, $(2,4)$, $(\frac{5}{2},\frac{25}{4})$, $(3,9)$.

2) Have them note that the graph of $y = x^2$, where $x \geq 0$ appears to be a smooth curve such as appears in the figure at the right:



b. How can we read $\sqrt{3}$ from the graph? We draw the line $y = 3$ and find at which point it crosses the graph of $y = x^2$. Then the x-coordinate of this point is $\sqrt{3}$. It is read approximately as 1.7.

c. Have pupils read $\sqrt{4}$, $\sqrt{5}$, $\sqrt{6}$, $\sqrt{7}$, $\sqrt{8}$, $\sqrt{9}$ from the graph. Have them check their results by reference to a table of square roots.

B. Suggested Practice

1. Between which two consecutive integers is each of the following:

   a. $\sqrt{5}$ Solution: Between 2 and 3, or $2 < \sqrt{5} < 3$

   b. $\sqrt{31}$ c. $\sqrt{67}$ d. $\sqrt{85}$ e. $\sqrt{137}$ f. $\sqrt{219}$

2. For any positive integer less than 100, how many digits are there in the integral part of its square root?

3. What is the smallest positive integer whose square root is a two-digit integer?

4. Find the indicated square root to the nearest tenth:

   a. $\sqrt{32}$ d. $\sqrt{111}$

   b. $-\sqrt{17}$ e. $\sqrt{6.25}$

   c. $\sqrt{9.6}$ f. $-\sqrt{20.9}$

5. Check the results in 2a and 2b by using a table of square roots.

6. Use a table of squares to find the squares of:

   20 22 23 78 99

7. Use a table of square roots to find, correct to hundredths place:

   $\sqrt{5}$ $\sqrt{8}$ $\sqrt{18}$ $\sqrt{70}$ $\sqrt{94}$

8. Which of the following numbers will have square roots that are:

   a. less than 10?

   b. greater than 10, but less than 100?

   c. greater than 100 but less than 1000?

   36 121 6400 16,900 490,000

   72 385 5238 22,850 810,000

9. Find, to the nearest tenth, the square root of the number which is the sum of the squares of 4 and 9.

10. Find, to the nearest tenth, the side of a square whose area is 29 square inches.

11. Find both roots of each of the following to the nearest tenth:

$a^2 = 360$ $b^2 = 22.6$

VI. The Pythagorean Theorem

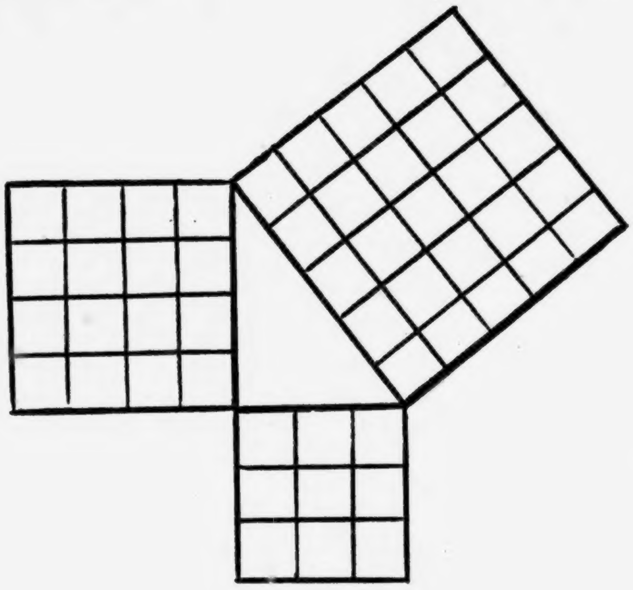
A. Suggested Procedure

1. Have pupil learn how the ancient Egyptian surveyors and engineers (rope-stretchers) laid out square corners by means of knotted ropes (3, 4, 5 triangle).

2. Review meaning of right triangle, hypotenuse, legs of right triangle.

3. Help pupils discover the relationship that exists among the three sides of a right triangle.

   a. Have them use squared paper in the construction of a right triangle whose legs are 3 and 4 units respectively. Have them build squares on each side. Have them measure the hypotenuse with a strip of the squared paper. Have pupils note that $(3)^2 + (4)^2 = 25$.



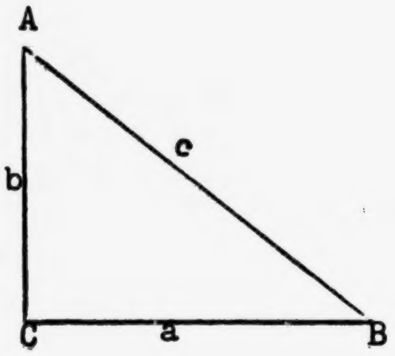
   b. Have pupils repeat this procedure with right triangles whose legs are 6, 8; 5, 12; etc. Have pupils denote the hypotenuse as c and the other two sides as a and b. Have them record the results of the experiment in tabular form. Have them make a similar table for the squares of the sides.

a	b	c
6	8	10
5	12	13
8	15	17
etc.		

a^2	b^2	c^2
36	64	100
etc.		

Have pupils note that $a^2 + b^2 = c^2$.

c. Have pupils realize that in any right triangle, the square of the length of the hypotenuse equals the sum of the squares of the lengths of the other two sides.



$$a^2 + b^2 = c^2$$

d. Tell pupils that this relationship is known as the Pythagorean Theorem.

Have them realize that they have not proved the Pythagorean Theorem, but have only examined some illustrations of it. (It will be proved in 10th Year Mathematics.) The first proof of the theorem is credited to Pythagoras, a mathematician of the 6th century B.C.

e. Inform pupils that it can also be proved that if a, b and c designate the lengths of the three sides of a triangle, with a and b less than c, then the triangle is a right triangle if

$$a^2 + b^2 = c^2$$

This gives us a way of determining whether a triangle, the lengths of whose sides are known, is a right triangle.

4. Help pupils solve problems using the Pythagorean Theorem. (Use numbers whose square roots can be readily determined.)

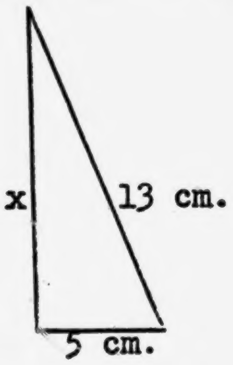
a. The legs of a right triangle are 9 inches and 12 incnes. What is the length of the hypotenuse?

$$\begin{aligned} a^2 &+ b^2 = c^2 \\ 9^2 &+ 12^2 = c^2 \\ 81 &+ 144 = c^2 \\ &225 = c^2 \\ &15 = c \end{aligned}$$

Check:
$$\begin{aligned} 9^2 + 12^2 &\overset{?}{=} 15^2 \\ 81 + 144 &\overset{?}{=} 225 \\ 225 &= 225 \end{aligned}$$

The length of the hypotenuse is 15 inches. (Here we consider only the principal square root since our problem cannot have a negative answer.)

b. The hypotenuse of a right triangle is 13 cm. long. One leg is 5 cm. long. What is the length of the other?



$$\begin{aligned} a^2 &+ b^2 = c^2 \\ 5^2 &+ x^2 = 13^2 \\ 25 &+ x^2 = 169 \\ &x^2 = 144 \\ &x = 12 \end{aligned}$$

Check:
$$\begin{aligned} 5^2 + 12^2 &\overset{?}{=} 13^2 \\ 25 + 144 &\overset{?}{=} 169 \\ 169 &= 169 \end{aligned}$$

The length of the other leg is 12 cm.

B. Suggested Practice

Note: Some measures of lengths of sides of right triangles are:

3, 4, 5 and any multiples of these

5, 12, 13; 8, 15, 17; 7, 24, 25; 20, 21, 29; 9, 40, 41

and any multiples of these.

1. Find the length of the missing side

	Leg	Leg	Hypotenuse
a.	?	12	15
b.	7	24	?
c.	15	?	17

2. Illustrate each of the following problems by a diagram before solving:

   a. The bottom of a valise measures 18" x 24". What is the maximum length of an umbrella which can be placed on the bottom of the valise?

   b. What is the length of a diagonal brace for an iron gate 24" long and 27" wide?

   c. A pole 40 ft. in height is steadied by a guy wire 41 ft. in length. How far from the foot of the pole is the foot of the guy wire?

   d. The length of a rectangle is twice its width. If the diagonal of the rectangle is 10 feet, what are its length and width? (to the nearest tenth)

   e. How long is the diagonal of a square whose side is 1 inch? Leave answer as a radical.

3. (OPTIONAL) Which of the following sets of numbers may be the lengths of sides of a right triangle?

   a. 3, 4, 5

   Solution: If $a = 3$, $b = 4$, $c = 5$, does $a^2 + b^2 = c^2$?

   $$3^2 + 4^2 \overset{?}{=} 5^2$$

   $$9 + 16 \overset{?}{=} 25$$

   $$25 = 25$$

   Then 3, 4, 5 may be the lengths of the sides of a right triangle.

   b. 8, 10, 15

   c. 14, 48, 50

   d. 5, 8, 11

   Note: A triple of positive integers (a, b, c) is said to be a Pythagorean triple if $a^2 + b^2 = c^2$. However, almost all right triangles have at least one irrational side.

VII. Visualizing Irrational Square Roots as Points on a Number Line

A. Suggested Procedure

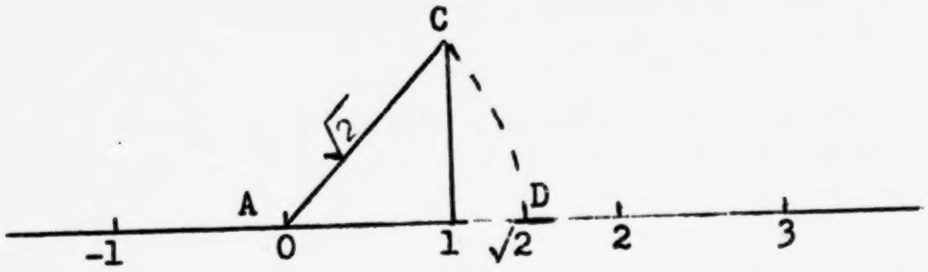
1. Review locating rational approximations on the number line for irrational square roots. (See Chapter XI)

2. Locating irrational square roots on the number line without using approximations.

a. Pose question: How can we find a point on the number line to associate with $\sqrt{2}$, without using approximations?

b. Demonstrate locating $\sqrt{2}$ on the number line as follows:

By constructing a right triangle whose legs are both 1, we can show that the hypotenuse = $\sqrt{2}$



Since AC equals $\sqrt{2}$, we use the length of AC as a radius and A as the center to find the point D; then AD is the length of $\sqrt{2}$ and D corresponds to the point which is at a distance of $\sqrt{2}$ from zero. Similarly, many other irrational numbers can be graphed on the number line.

B. Suggested Practice

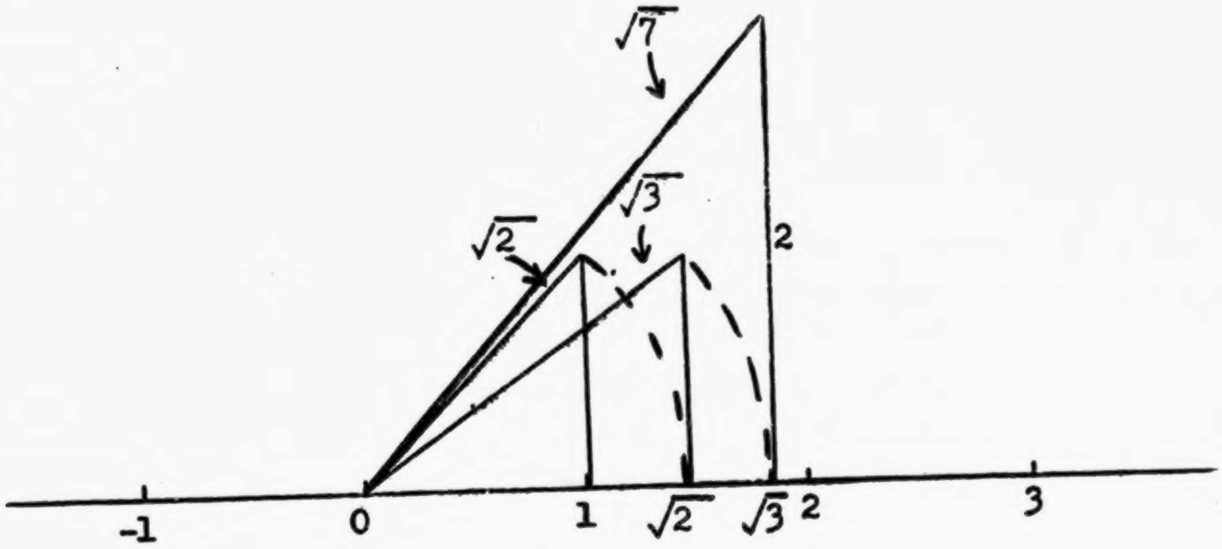
1. Place $\sqrt{8}$, $\sqrt{10}$, $\sqrt{13}$ on the number line without using approximations.

Hint: For $\sqrt{8}$, legs are 2, 2

For $\sqrt{10}$, legs are 1, 3

For $\sqrt{13}$, legs are 2, 3

2. (OPTIONAL) Locate the following square roots as points on the number line without using approximations: $\sqrt{3}$, $\sqrt{7}$



VIII. Simplification of Radicals

A. Suggested Procedure

1. Square roots of numeral products

a. Have pupils recall that the square root table provided in their textbook gives the square roots of perfect squares exactly, and rational approximations for the square roots of other integers.

Pose question: Although the table is often limited to integers from 1 to 100, how can it be used for other integers as well?

b. Have pupils consider the following products of square roots:

$\sqrt{9}\cdot\sqrt{4} = 3\cdot 2$, or 6 $\qquad$ $\sqrt{36} = 6$

$\sqrt{4}\cdot\sqrt{16} = 2\cdot 4$, or 8 $\qquad$ $\sqrt{64} = 8$

$\sqrt{16}\cdot\sqrt{9} = 4\cdot 3$, or 12 $\qquad$ $\sqrt{144} = 12$

Have them realize that

$$\sqrt{9}\cdot\sqrt{4} = \sqrt{36} = \sqrt{9\cdot 4}$$

$$\sqrt{4}\cdot\sqrt{16} = \sqrt{64} = \sqrt{4\cdot 16}$$

$$\sqrt{16}\cdot\sqrt{9} = \sqrt{144} = \sqrt{16\cdot 9}$$

c. Help pupils reach a generalization concerning the product of square roots.

1) Have them suggest the following symbolic statement concerning the product of square roots.

$\sqrt{a}\cdot\sqrt{b} = \sqrt{ab}$, or $\sqrt{ab} = \sqrt{a}\cdot\sqrt{b}$, where $a > 0$ and $b > 0$.

2) By trying various replacements for a and b have pupils see that this appears to be a true statement for every replacement of the variables a and b by a non-negative real number.

3) Have pupils suggest the following verbal brief statement. Roots are distributed over multiplication. Have them contrast with $\sqrt{a+b}$.

$\sqrt{9+4} \neq 3+2$ and therefore roots are not distributed over addition.

d. Have pupils understand how the above generalization can be used to facilitate finding the square roots of many numbers not included in a square root table.

1) Find $\sqrt{200}$

$\sqrt{200} = \sqrt{100 \cdot 2} = \sqrt{100} \cdot \sqrt{2}$

$\sqrt{200} = 10\sqrt{2}$

Referring to the square root table, we find $\sqrt{2} \approx 1.73$

Then $\sqrt{200} \approx 10 \times 1.73$, or 17.3.

2) Tell pupils that $10\sqrt{2}$ is said to be the simplest form of expressing $\sqrt{200}$ because the new radicand 2 has no factor, other than 1, which is a perfect square.

e. Guide pupils to see how the square root of every positive integer can be expressed in simplest form (as an integer, or as the product of an integer and the square root of an integer having no square factor other than 1).

1) Simplify $\sqrt{576}$

$576 = 4 \cdot 144$ and $\sqrt{576} = \sqrt{4 \cdot 144}$

Then $\sqrt{576} = \sqrt{4} \cdot \sqrt{144} = 24$ (Simplest form)

2) Simplify $\sqrt{75}$

$75 = 25 \cdot 3$ and $\sqrt{75} = \sqrt{25 \cdot 3}$

$\sqrt{75} = \sqrt{25} \cdot \sqrt{3} = 5\sqrt{3}$ (Simplest form)

3) Simplify $\sqrt{250}$ and then approximate it by use of a table of square roots.

$250 = 25 \cdot 10$ and $\sqrt{250} = \sqrt{25 \cdot 10}$

$\sqrt{250} = \sqrt{25} \cdot \sqrt{10} = 5\sqrt{10}$ (Why is this the simplest form?)

Referring to the table, $\sqrt{10} \approx 3.16$

Then, $\sqrt{250} \approx 5 \times 3.16$, or 15.80.

4) Simplify $\sqrt{48}$

$48 = 16\cdot 3$

$\sqrt{48} = \sqrt{16}\cdot\sqrt{3} = 4\sqrt{3}$

Suppose we had written $48 = 4\cdot 12$ and $\sqrt{48} = \sqrt{4}\cdot\sqrt{12}$ or $2\sqrt{12}$.

Is this the simplest form of $\sqrt{48}$? (No, because 12 has a factor, 4, which is a perfect square.) Then to complete the work, we write:

$\sqrt{48} = \sqrt{4}\cdot\sqrt{12} = 2\sqrt{12} = 2\sqrt{4}\cdot\sqrt{3} = 2\cdot 2\sqrt{3} = 4\sqrt{3}$

Note: A similar method can be used to find square roots of numbers which are not integers, e.g.,

$\sqrt{.2}$, $\sqrt{.03}$, etc.

f. Have pupils express the following square roots in simplest form:

1) $\sqrt{8}$
2) $\sqrt{50}$
3) $\sqrt{98}$
4) $\sqrt{160}$
5) $\sqrt{245}$
6) $\sqrt{2025}$
7) $\sqrt{116}$
8) $\sqrt{720}$
9) $\sqrt{.04}$

Have them use a square root table to approximate 4), 5), 7), and 8) above.

2. Square roots of monomial products (The assumption is that each letter in the radicand represents a non-negative number.)

a. Have pupils consider

$x\cdot x = x^2$ and $\sqrt{x^2} = x$

$x^2\cdot x^2 = x^4$ and $\sqrt{x^4} = x^2$

$x^3\cdot x^3 = x^6$ and $\sqrt{x^6} = x^3$ $\quad x \geq 0$

Have them realize that the above sentences are true statements for every replacement of x by a non-negative real number.

b. Simplify $\sqrt{25x^2}$

$\sqrt{25x^2} = \sqrt{25}\cdot\sqrt{x^2} = 5x$ (Simplest form)

Check: $5x\cdot 5x = 25x^2$

c. Simplify $\sqrt{x^2y^4}$

$\sqrt{x^2y^4} = \sqrt{x^2} \cdot \sqrt{y^4} = xy^2$ (Simplest form)

d. Simplify $\sqrt{2x^2}$

$\sqrt{2x^2} = \sqrt{2} \cdot \sqrt{x^2} = \sqrt{2}x$ (Simplest form)

e. Simplify $\sqrt{a^3}$

$\sqrt{a^3} = \sqrt{a^2} \cdot \sqrt{a} = a\sqrt{a}$ (Simplest form)

f. Simplify $\sqrt{m^5}$

$\sqrt{m^5} = \sqrt{m^4} \cdot \sqrt{m} = m^2\sqrt{m}$

Check: $(m^2\sqrt{m})^2 = (m^2)^2 \cdot (\sqrt{m})^2 = m^4 \cdot m$ or m^5

g. Simplify $\sqrt{32b^3}$

$\sqrt{32b^3} = \sqrt{4b^2} \cdot \sqrt{8b}$

$= 2b\sqrt{8b}$ (Not in simplest form)

$= 2b\sqrt{4} \cdot \sqrt{2b}$

$= 2b \cdot 2\sqrt{2b}$ or $4b\sqrt{2b}$

Lead pupils to understand that by selecting the largest perfect square as a factor of the radicand, he reduces the number of steps necessary for simplification. Thus,

$\sqrt{32b^3} = \sqrt{16b^2} \cdot \sqrt{2b}$

$= 4b\sqrt{2b}$

Check: $(4b\sqrt{2b})^2 = (4b)^2 \cdot (\sqrt{2b})^2 = 16b^2 \cdot 2b = 32b^3$

B. Suggested Practice

1. Simplify the following:

a. $\sqrt{441}$ $\sqrt{1089}$ $\sqrt{1764}$ $\sqrt{1225}$ $\sqrt{8100}$

b. $\sqrt{b^6}$ $\sqrt{c^8}$ $\sqrt{x^4}$ $\sqrt{a^4b^8}$

c. $\sqrt{256x^4}$ $\sqrt{36m^6}$ $\sqrt{324a^8}$

d. $\sqrt{\frac{36}{81}x^2}$ $\sqrt{\frac{64}{25}y^4}$ $\sqrt{\frac{81}{100}x^4}$ $\sqrt{\frac{121a^5}{169}}$

e. $\sqrt{.09b^2}$ $\sqrt{49x^2y^2}$ $\sqrt{.36a^2b^2}$ $\sqrt{.0064x^2}$ $\sqrt{.0081y^6}$

f. $\sqrt{128}$ Solution: $\sqrt{128} = \sqrt{64} \cdot \sqrt{2} = 8\sqrt{2}$

$\sqrt{18}$ $\sqrt{32}$ $\sqrt{50}$ $\sqrt{12}$ $\sqrt{14}$ $\sqrt{48}$ $\sqrt{125}$ $\sqrt{98}$ $\sqrt{2048}$

g. $\sqrt{x^3}$ $\sqrt{y^5}$ $\sqrt{m}$ $\sqrt{a^3x^9}$ $\sqrt{x^6y^3}$ $\sqrt{a^3b^4}$

h. $\sqrt{8a}$ $\sqrt{18b^3}$ $\sqrt{60c^3}$ $\sqrt{12x^5}$ $\sqrt{288m^7}$ $\sqrt{49a^3}$ $\sqrt{12x^4}$

i. The sides of a right triangle are $\sqrt{6}$ and $\sqrt{6}$. What is the length of the hypotenuse? (Leave answer in simplest radical form.)

j. The sides of a right triangle are $\sqrt{80}$ and $\sqrt{60}$. What is the length of the hypotenuse to the nearest tenth?

CHAPTER XII

QUADRATIC EQUATIONS

This chapter contains materials and suggested procedures for reinforcing and extending the pupils' ability to solve quadratic equations in one variable, and to apply these algebraic skills to the solution of various verbal problems.

I. Incomplete Quadratic Equations

A. Suggested Procedure

1. Review:

    a. Degree of an equation

    b. Zero product. If $a \cdot b = 0$, then $a = 0$, or $b = 0$, or a and $b = 0$.

    c. Every non-negative number except zero has two real square roots, which are additive inverses of each other. (The square root of zero is zero, and zero is its own additive inverse.

2. Meaning of incomplete quadratic equations

    Note: The word quadratic comes from the Latin, "quadratus" meaning "squared". It is used in connection with the second-degree equation because the highest exponent is 2 - the variable squared.

    a. Have pupils compare the following equations:

    1) $x^2 + 3x + 2 = 0$

    2) $x^2 + 2 = 0$

    How are they alike? (Each is a second-degree or quadratic equation. The second-degree terms are the same, as are the constant terms.)

    How are they different? (There is no first-degree term in equation 2.)

    b. Tell pupils that a quadratic equation is said to be incomplete if the linear or first degree term is missing.

    c. Have pupils identify the incomplete quadratic equations among the following:

    1) $x^2 - 49 = 0$ 3) $a^2 = a + 6$

    2) $x^2 + 9x + 8 = 0$ 4) $2y^2 = 50$

3. Solving incomplete quadratic equations

a. Have pupils recall how an equation such as $x^2 - 25 = 0$ was solved by factoring.

$$x^2 - 25 = 0$$

$$(x-5)(x+5) = 0$$

Then $x-5 = 0$ or $x+5 = 0$ Why?
$x = 0$ $x = -5$

The solution set is $\left\{5, -5\right\}$.

Check:

$$5^2 - 25 \overset{?}{=} 0 \qquad (-5)^2 - 25 \overset{?}{=} 0$$

$$25 - 25 \overset{?}{=} 0 \qquad 25 - 25 \overset{?}{=} 0$$

$$0 = 0 \qquad 0 = 0$$

b. Have pupil see how the foregoing equation could have been solved differently, as follows:

$$x^2 - 25 = 0$$

$$x^2 = 25 \quad \text{(Equivalent equation)}$$

Then, since every non-negative number, except 0, has two real square roots which are additive inverses of each other,

$$x = \sqrt{25} \text{ or } 5, \text{ or } x = -\sqrt{25} \text{ or } -5$$

The solution set is $\left\{5, -5\right\}$.

c. Have pupils use the method in b to solve the following equations:

1) Solve $3a^2 - 21 = 0$

$$3a^2 = 21$$

$$a^2 = 7$$

$$a = \sqrt{7} \text{ or } a = -\sqrt{7}$$

The solution set is $\left\{\sqrt{7}, -\sqrt{7}\right\}$.

Check: $3(\sqrt{7})^2 - 21 \overset{?}{=} 0$ $\qquad$ $3(-\sqrt{7})^2 - 21 \overset{?}{=} 0$

$$3 \cdot 7 - 21 \overset{?}{=} 0 \qquad 3 \cdot 7 - 21 \overset{?}{=} 0$$

$$0 = 0 \qquad 0 = 0$$

2) Solve $(y + 5)^2 = 36$

Although this is not an incomplete quadratic equation as could be determined by squaring the left member, the form of this equation is like that of an incomplete quadratic.

Since the square of an expression is equal to a number, then the expression is equal to one of the two square roots of the number.

$$\begin{array}{rclcrcl} y + 5 &=& \sqrt{36} & \text{or} & y + 5 &=& \sqrt{36} \\ y + 5 &=& 6 & & y + 5 &=& -6 \\ y &=& 1 & & y &=& -11 \end{array}$$

The solution set is $\{1, \ -11\}$.

Check:

$$\begin{array}{rclcrcl} (1+5)^2 &\overset{?}{=}& 36 & \qquad & (-11+5)^2 &\overset{?}{=}& 36 \\ 6^2 &\overset{?}{=}& 36 & & (-6)^2 &\overset{?}{=}& 36 \\ 36 &=& 36 & & 36 &=& 36 \end{array}$$

3) Solve $x^2 + 4 = 0$

$x^2 = -4$ (Equivalent equation)

Have pupils recall that negative numbers have no square roots in the set of real numbers.

Have them conclude that in the system of real numbers, the solution set of the above equation is the null set.

d. Solving formulas for a variable that is squared

1) Have pupils find the length of a side of a square when the area is 9 square inches.

$$A = s^2$$

$$9 = s^2$$

$\pm\sqrt{9} = s$, i.e., $s = 3$, $s = -3$

The nature of the problem requires that the domain of s be limited to the non-negative numbers. Therefore, the solution set is $\{3\}$.

2) Solve $A = s^2$ for s (Area of a square) $\sqrt{A} = s$ or $-\sqrt{A} = s$

Why do we not use the negative root?

The solution set is $\left\{\sqrt{A}\right\}$.

Check: $A \overset{?}{=} (\sqrt{A})^2$

$A = A$

3) Solve $W = \frac{E^2}{R}$ for E (Formula for electrical power)

$WR = E^2$ (Why?)

$\sqrt{WR} = E$ or $-\sqrt{WR} = E$ (Why?)

The solution set is $\left\{\sqrt{WR}\right\}$. Have pupils check.

B. Suggested Practice

Solve and check each of the following:

If any root is irrational, express it in simplest radical form. Check.

1. $y^2 - 1 = 0$
2. $(y+3)^2 = 16$
3. $x^2 = \frac{1}{4}$
4. $a^2 - .09 = 0$
5. $2x^2 - 16 = 0$
6. $x^2 = 50$
7. $25n^2 = 4$
8. $(z-6)^2 = 49$
9. $(a-7)^2 = 100$
10. $(b+5)^2 = 81$
11. $(x+.3)^2 = .04$

12. What is the solution set of $x^2 + 36 = 0$ in the system of real numbers? Explain.

13. Solve $A = 6e^2$ for e (Formula for area of the surface of a cube)

14. Find approximate values for all irrational roots in Exercises 1-11. (Use a square root table.)

II. Complete Quadratic Equations

A. Suggested Procedure

1. Have pupils solve the following complete quadratic by factoring:

Solve $x^2 - 2x + 1 = 4$

$$x^2 - 2x - 3 = 0$$

$$(x-3)(x+1) = 0$$

Then, $x - 3 = 0$ or $x + 1 = 0$
$x = 3$ $\quad$ $x = -1$

Check: $(3)^2 - 2(3) + 1 \overset{?}{=} 4$ $\qquad$ $(-1)^2 - 2(-1) + 1 \overset{?}{=} 4$

$9 - 6 + 1 \overset{?}{=} 4$ $\qquad$ $1 + 2 + 1 \overset{?}{=} 4$

$4 = 4$ $\qquad$ $4 = 4$

2. Solution by taking square roots

$$x^2 - 2x + 1 = 4$$

Since the left member is a trinomial square, we may express it as the square of a binomial.

$$(x-1)^2 = 4$$

Then, $x - 1 = 2$ or $x - 1 = -2$

$x = 3$ $\qquad$ $x = -1$

Why are we able to use the square root method to solve the above equation? (The left member is a trinomial square and the right member is a positive number.)

3. Solution of complete quadratic equations by "completing the square"

a. Have pupils consider the equation $x^2 - 2x = 8$.

Can this be solved by the factoring method?

Have pupils solve it by factoring. They determine the solution set is $\{4, -2\}$.

b. Pose question:

Can $x^2 - 2x = 8$ be solved by taking square roots?

Have pupils realize that inasmuch as the left member is not a trinomial square, we cannot make use of this method. If, however, we can find an equation which is equivalent to $x^2 - 2x = 8$, but which has a trinomial square as one member, we can easily find the solution set by taking square roots.

1) Pose question:

What shall we add to $x^2 - 2x$ to produce a trinomial square?

$x^2 - 2x + ? =$ a trinomial square

By trial and error, pupils will arrive at $x^2 - 2x + \underline{1}$.

Then, $x^2 - 2x + \underline{1} = 8 + \underline{1}$ (Addition principle of equations)

or, $(x-1)^2 = 9$.

2) Have pupils now use the square root method to complete the solution.

$x - 1 = 3$ or $x - 1 = -3$

$x = 4$ $\qquad$ $x = -2$

The solution set is $\{4, -2\}$.

c. Tell pupils that the process of starting with the polynomial $x^2 - 2x$ and producing the trinomial square $x^2 - 2x + 1$ is called completing the square.

d. Have pupils realize that it is not always easy to tell just what number should be added in order to obtain a trinomial square. Therefore, a systematic method for finding this number is needed.

Have them examine the following to see the relationship between the coefficient of the first degree term and the constant term, when the coefficient of the second degree term is 1.

1) $(x+3)^2 = x^2 + 2(3)x + 9 = x^2 + 6x + 9$

They note: 3 is $\frac{1}{2}$ of 6 and $9 = 3^2$

2) $(x-5)^2 = x^2 + 2(-5)x + (-5)^2 = x^2 - 10x + 25$

They note: -5 is $\frac{1}{2}$ of -10 and $25 = (-5)^2$

3) $(a+4)^2 = a^2 + 2(4)a + 4^2 = a^2 + 8a + 16$

They note: 4 is $\frac{1}{2}$ of 8 and $16 = 4^2$

Guide pupils to see that in each case the constant term of the trinomial square is the square of half the coefficient of the first-degree term.

e. Have pupils complete the square for the following. Have them express the resulting trinomial as the square of a binomial.

1) $x^2 + 2x +$ _____

Solution: $\frac{1}{2}$ of 2 is 1 and $1^2 = 1$. Then we add 1 to $x^2 + 2x$ to produce $x^2 + 2x + 1$, a trinomial square.

$$x^2 + 2x + 1 = (x+1)^2$$

2) $y^2 - 4y +$ _____

3) $a^2 + 16a +$ _____

4) $b^2 - 12b +$ _____

5) $x^2 + x +$ _____

6) $r^2 - 5r +$ _____

7) $t^2 - \frac{1}{2}t +$ _____

f. Have pupils solve quadratic equations by completing the square.

1) Solve $x^2 - 4x - 12 = 0$

$x^2 - 4x = 12$ (Addition principle of equations)

$x^2 - 4x + 4 = 12 + 4$ (Complete the square)

$(x-2)^2 = 16$ (Express the trinomial as the square of a binomial)

Then, $x - 2 = 4$ or $x - 2 = -4$

$x = 6$ $\qquad x = -2$

The solution set is $\{6, -2\}$.

Have pupils check by determining whether the solutions satisfy the equation.

Note: The solution of a quadratic equation by the method of completing the square seems longer and more involved than the factoring method. However, the former is a more general method which applies also to quadratic trinomials not factorable in the system of rational numbers. Then, too, mathematical power is increased when pupils know more than one method of solution.

2) Solve $2x^2 - 4x = 6$

Have pupils realize that since the coefficient of x^2 is 2, the procedure that was developed for completing the square will not apply.

Guide them to see, however, that the given equation can be replaced by the equivalent equation.

$$x^2 - 2x = 3$$

This equivalent equation is solved instead of the original equation. Have pupils solve it by completing the square. Have them check the solutions in the original equation.

B. Suggested Practice

1. What number should be added to each of the following expressions to make it a trinomial square? Express the resulting trinomial as the square of a binomial.

   a. $x^2 - 2x$ Solution: Add 1. $x^2 - 2x + 1 = (x-1)^2$

   b. $b^2 + 6b$

   c. $y^2 + 12y$

   d. $a^2 - 10b$

   e. $y^2 - 20y$

   f. $t^2 + 3t$

   g. $x^2 - x$

   h. $r^2 + \frac{1}{2}r$

2. Solve the following equations by the method of completing the square.

   a. $y^2 - 6y = 7$

   b. $a^2 - 4a - 32 = 0$

   c. $x^2 + 2x - 48 = 0$

   d. $t^2 + t = 12$

   e. $2x^2 - 8x = 24$

   f. $x^2 = 10x + 144$

3. In solving the following problems, solve the equation for each problem by the method of completing the square.

   a. If 4 times a certain number is subtracted from the square of the number, the result is 25. Find the number or numbers that satisfy this condition.

   b. There are two consecutive positive integers such that the square of the first decreased by twice the second is 33. What are these integers?

   c. The hypotenuse of a right triangle is 10 inches long. The difference between the lengths of the other two sides is 2 inches. Find the lengths of the sides of this triangle.

   d. The difference between a number and its reciprocal is $\frac{3}{2}$. Find the number or numbers.

CHAPTER XIII

RATIO, PROPORTION AND INDIRECT MEASUREMENT

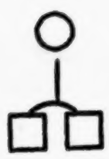
This section contains materials and suggested procedures for extending pupils' concepts of ratio and proportion, and for relating these concepts to the solution of problems in indirect measurement by means of similar triangles and the numerical trigonometry of the right triangle.

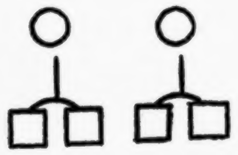
I. Ratio

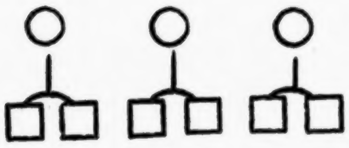
A. Suggested Procedure

1. Review

a. The meaning of ratio as the relation between the numbers of two sets.







In each of the above diagrams, 1 circle is matched with two squares. Thus, in each case, the set of circles is related to the set of squares as 1 is related to 2.

Also, the set of squares is related to the set of circles as 2 is related to 1. This kind of relationship between the numbers of two sets is called a ratio.

b. Ways of expressing ratio

In each of the above examples the ratio of the number of circles in each set of circles to the number of squares in each set of squares may be expressed as 1 to 2, or 1:2, or 1 - 2, or $\frac{1}{2}$, or 50%, or .5.

2. Extend the concept of ratio

a. Pose problem: An article in a school newspaper stated that the ratio of the number of girls to the number of boys in the Glee Club is 4 to 1. Can we tell from this information how many pupils are members of the Glee Club?

1) Elicit that the ratio 4 to 1 does not tell us how many pupils are in the club, but does tell us that the number of girls is four times the number of boys, regardless of the total number. That is to say, for every four girls in the club, there is one boy.

2) Ask such questions as the following:

If there are 5 members in the Glee Club, how many are girls? boys?
If there are 10 members in the Glee Club, how many are girls? boys?

3) Have pupils fill in the following chart based on the conditions stated in the problem:

NUMBER OF GIRLS AND BOYS IN GLEE CLUB

Girls	64	?	32	44	?	4n	?	?
Boys	?	15	?	?	x	?	2y	16y

4) Have the pupils form the ratio in each case and show that the ratio in each case remains 4 to 1.

$\frac{64}{16} = \frac{4\cdot16}{1\cdot16} = \frac{4}{1}\cdot1$, or $\frac{4}{1}$ $\qquad$ $\frac{60}{15} = \frac{4\cdot15}{1\cdot15} = \frac{4}{1}\cdot1$, or $\frac{4}{1}$

5) Have them see that the ratio $\frac{4}{1}$ may also be expressed by the number pairs $\frac{8}{2}$, $\frac{44}{11}$, $\frac{60}{15}$, $\frac{-20}{-5}$, and, in general, $\frac{4x}{x}(x\neq0)$.

Since $\frac{4}{1}$ uses the smallest integers possible, we say that $\frac{4}{1}$ is the simplest form of the ratio.

6) Have pupils recall that a ratio consists of two numbers in a definite order. Thus, the ratio of 4 to 5 is written $\frac{4}{5}$ or 4:5, whereas the ratio of 5 to 4 is written 5 to 4 or 5:4. A ratio may therefore be thought of as an ordered pair of numbers.

b. Help pupils generalize that the ratio $\frac{a}{b}$ may also be expressed as $\frac{ax}{bx}$ for any replacement of x other than zero. That is to say, all the pairs of numbers $\frac{ax}{bx}$ name the same ratio.

c. Have pupils express in terms of x two numbers whose ratio is

1) $\frac{3}{4}$ Solution: 3x and 4x
2) $\frac{5}{6}$
3) $\frac{8}{7}$
4) $\frac{1}{4}$

3. Have pupils learn to solve ratio problems using algebraic methods.

   a. Pose problem:

      An article in the school newspaper stated that the ratio of the number of girls in the Spanish Club to the number of boys is 5:2. If the Spanish Club consists of 28 members, how many are girls and how many are boys?

      1) Solution through arithmetic

         Help the pupils see that the number of boys and the number of girls depends upon the total number in the group. At the same time, these numbers must be in the ratio of 5 to 2.

         For every set of 5 girls, there is a set of 2 boys.

$$\begin{array}{ccccc} 5 & + & 2 & = & 7 \\ 5 & + & 2 & = & 7 \\ 5 & + & 2 & = & 7 \\ \underline{5} & + & \underline{2} & = & \underline{7} \\ 20 \text{ girls} & + & 8 \text{ boys} & = & 28 \text{ in club} \end{array}$$

         Elicit that there are 4 groups of each set, making a total of 28.

      2) Solution through algebra

         Discuss with the pupils difficulties one would have in solving a more complicated problem involving many groups, by this method.

         Tell the pupils that an algebraic solution would be more direct and simpler. In some cases it provides the only method of solution.

         Help the pupils to see that since the ratio of girls to boys is 5 to 2, we can represent the number of girls by $5x$, and the number of boys by $2x$, $x \neq 0$. It is obvious from the problem that the domain of x is the set of positive integers.

$$\begin{aligned} \text{Let } 5x &= \text{the number of girls} \\ \text{Let } 2x &= \text{the number of boys} \\ 5x + 2x &= 28 \text{ (the total number of pupils is 28)} \\ 7x &= 28 \\ x &= 4;\ 5x = 20;\ 2x = 8 \end{aligned}$$

         The number of girls is 20. The number of boys is 8.

         Check: Is the ratio of girls to boys 5 to 2? Yes, because $\frac{20}{8} = \frac{5}{2}$. Is the total number of pupils 28? Yes, because $20 + 8 = 28$.

b. Have pupils practice similar examples. (See suitable textbooks.)

B. Suggested Practice

1. Express each of the following ratios in its simplest form:

   a. 2 to 4

   b. 9:27

   c. .5 to 1.5

   d. $\frac{3x}{3y}$

   e. $\frac{ax}{ay}$

   f. x to 4x

   g. $\frac{2a^3}{a^3}$

   h. $9t:27t^3$

   i. $\frac{.4x^3y^2}{.8xy^4}$

   j. $\frac{1}{2}xg:\frac{1}{4}xy^2$

2. Find the ratio of each of the following. Express each answer in simplest form.

   a. The number of boys to the number of girls in a school with 700 boys in 1100 pupils.

   b. The number of pounds of sand to the number of pounds of clay if there are 25 pounds of sand in a mixture of both weighing 225 pounds.

   c. The number of pennies to the number of nickels in a coin bank containing only these coins, with 15 pennies to 30 coins.

   d. The number of apples to the number of oranges in a bag of fruit if x represents the number of apples ($x \neq 0$) and there are twice as many oranges.

   e. In a classroom the ratio of the number of boys to the number of girls is 2:3. What is the ratio of the number of boys to the total number of pupils? the number of girls to the total number of pupils?

3. Solve each of the following, and check:

   a. Helen invited 12 friends to her party. If the ratio of boys to girls was 1 to 3, how many boys did she invite? How many girls did she invite?

   b. Two numbers are in a ratio of 3 to 5. Their sum is 72. What are the numbers?

c. The ratio of the number of boys to the number of girls in the 9th grade is 7:8. The total number of pupils enrolled in our 9th grade is 540. How many are boys? How many girls are there?

d. Divide a 15 inch line segment into parts with a ratio of 2:3. How many inches should there be in each part of the line segment?

e. See suitable textbooks for additional similar problems.

II. Proportion

A. Suggested Procedure

1. Review meaning of proportion. A proportion is the equality of two ratios, e.g., $\frac{1}{2} = \frac{2}{4}$.

a. Pose problem:

John has a snapshot 4" wide and 5" long. He had an enlarged photograph made from it. The enlarged picture measured 8" in width and 10" in length.

1) What is the ratio of the width to the length of the original snapshot?

$$\frac{\text{Number of inches in width of snapshot}}{\text{Number of inches in length of snapshot}} = \frac{4}{5}$$

2) What is the ratio of the width to the length of the enlargement?

$$\frac{\text{Number of inches in width of enlargement}}{\text{Number of inches in length of enlargement}} = \frac{8}{10}$$

b. Elicit that the ratio of 8 to 10 is equal to the ratio of 4 to 5.

c. Have pupils recall that the equality of two ratios forms a proportion.

$\frac{4}{5} = \frac{8}{10}$ is a proportion and is read "4 is to 5 as 8 is to 10."

(4, 5, 8, and 10 are the first, second, third and fourth terms of the proportion respectively)

2. Have pupils find the value of a term of a proportion when the other three terms are known.

a. Pose problem:

John found that he wanted an enlargement of his 4" x 5" snapshot to measure 20" in length. What would be its width?

1) The ratio of the width to the length of the negative is $\frac{4}{5}$.

2) If we represent the number of inches in the width by x, what is the ratio of the width to the length of the enlargement? $(\frac{x}{20})$

3) Pupils should be led to realize that the ratio of the width to the length is the same in each case. This may be expressed as a proportion

$$\frac{4}{5} = \frac{x}{20}$$

$$16 = x$$

The width of the enlargement is 16".

4) **Check**: If the enlargement has a width of 16" and a length of 20", the ratio $\frac{16}{20}$ is equal to the ratio $\frac{4}{5}$ of the dimensions of the snapshot.

b. Have pupils practice with similar problems.

B. Suggested Practice

1. Find all values of the variable for which each proportion is true.

a. $\frac{x}{11} = \frac{24}{132}$

b. $\frac{19}{2} = \frac{95}{y}$

c. $\frac{24}{5} = \frac{x}{\frac{1}{4}}$

d. $\frac{17}{6} = \frac{y}{4}$

e. $\frac{x-2}{x} = \frac{3}{5}$

2. Solve and check.

a. The width of a rectangle is in the ratio of 5:7 to its length. If the length is 21 feet, what is the width?

b. A car travels 9 miles in 10 minutes. At that rate, how far will the car travel in 45 minutes?

Solution: Let x represent the number of miles the car will travel in 45 minutes. Then,

$$\frac{9}{x} = \frac{10}{45} \quad \text{or} \quad \frac{9}{10} = \frac{x}{45}$$

$$\begin{aligned} 10x &= 405 \\ x &= 40.5 \end{aligned}$$

The car will travel 40.5 miles in 45 minutes.

c. The ratio of the number of boys to the number of girls in a school is 9 to 10. There are 720 boys. How many girls are there in the school?

d.Dan's father drove his car 315 miles in 7 hours on Monday. On Tuesday he hopes to travel 405 miles at the same rate of speed. How many hours will he travel on Tuesday?

e. If 78 feet of wire weigh 13 lbs., what will 234 feet of the same kind of wire weigh?

f. A $1\frac{1}{4}$" line segment on a map represents a distance of 350 miles. What distance is represented by a 4" line segment?

III. Direct Variation

A. Suggested Procedure

1. Have pupils learn the meaning of direct variation.

a. Pose problem:

A train is traveling at a uniform rate of 40 m.p.h. How many miles will the train travel in 1 hour? 2 hours? 3 hours? 4 hours? 5 hours?

Distance	Time
40	1
80	2
120	3
160	4
200	5

Have pupils set up the table of values shown at the right.

Have pupils realize that each number pair in the table ($\frac{40}{1}, \frac{80}{2}, \frac{120}{3}, \frac{200}{5}$) is a name for the same ratio. That is to say in each case $\frac{d}{t} = 40$, or $d = 40t$. A relationship such as this is called a *direct variation*. We say that *d varies directly as t*. In the relationship $d = 40t$, the value of d is always 40 times the value of t: Thus, in this rule, 40 is called the *constant of variation*.

b. Have pupils repeat the procedure with the following:

The cost of one candy bar is 5¢. Two such candy bars cost 10¢, three such candy bars 15¢, etc.

Lead the pupils to express the ratio of cost to number bought as $\frac{c}{n}$. Then, in each case, $\frac{c}{n} = 5$ (a constant ratio), or $c = 5n$. Elicit that the total cost of the candy bars varies directly with the number bought, that is, c varies directly as n. What is the constant of variation in $c = 5n$?

c. Have pupils study the table at the right for the perimeter of an equilateral triangle.

s (side)	1	2	3	4	5
p (perimeter)	3	6	9	12	15

1) What relationship exists between the values of s and the corresponding values of p?

2) How would you express the ratio of each value of p to the corresponding value of s?

3) Is the ratio $\frac{p}{s}$ the same for each number pair?

4) Write an equation to express the relationship between p and s. $(p = 3s)$

5) What is the constant of variation?

2. Lead pupils to the generalization that two quantities x and y vary directly if they are related by an equation.

$$y = kx$$

where k is a constant (other than zero). We say "y varies directly as x." k is then called the constant of variation.

B. Suggested Practice

1. State whether each of the following tables expresses direct variation. If it does, determine the constant of variation and then write the equation that expresses this relation.

a.

s	1	3	7	9
p	4	12	28	36

b.

x	4	5	8	11
y	2	$2\frac{1}{2}$	4	$5\frac{1}{2}$

c.

a	2	3	4	5
b	1	2	3	4

2. Given that y varies directly as x, determine the constant of variation in each case and complete the table.

a.

x	1	2	3	?	9	?
y	4	8	?	24	?	48

b.

x	1	2	3	?	10
y	$\frac{1}{2}$	1	?	2	?

3. If y varies directly as x, and if y = 10, when x = 2, what is the value of y, when x = 7?

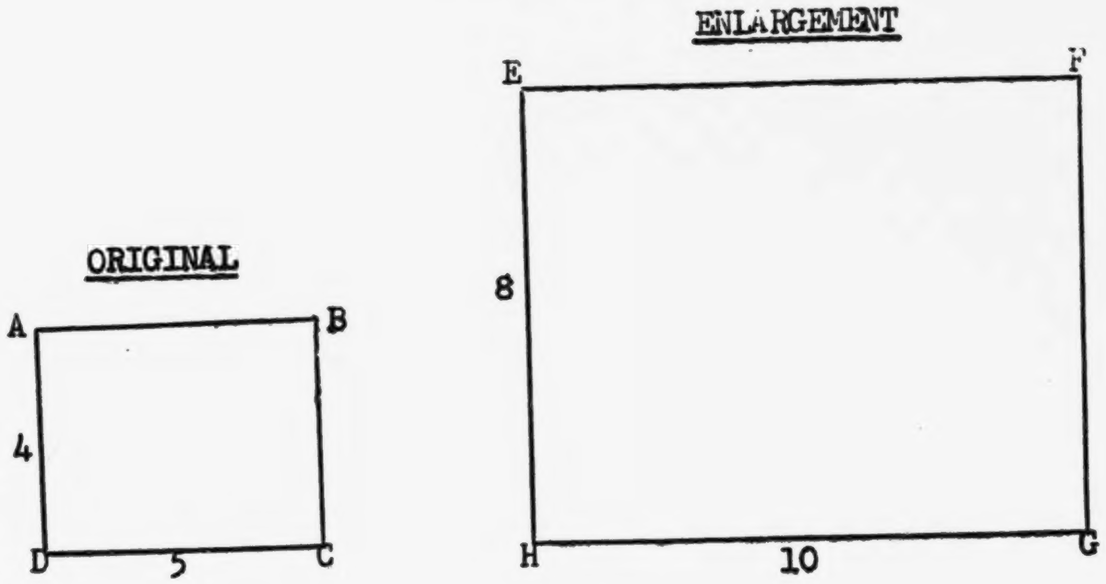
Solution: Since y varies directly as x, $y = kx$. When $y = 10$ and $x = 2$, we have $10 = k(2)$, or $k = 5$. The constant of variation is 5 and we may now write the original relationship as $y = 5x$. Then, when $x = 7$, $y = 5(7)$, or 35.

Note: Many scientific laws are expressed as the direct variation of two variables, e.g., Hooke's Law: $S = kw$ (The stretch of a spring varies directly as the weight suspended from it.)

IV. Similar Figures

A. Suggested Procedure

1. Have pupils recall the problem concerning the enlargement of a photograph. Help them represent the photograph and its enlargement, by means of the following rectangle diagrams:



In the two rectangles, vertex A is paired with or corresponds to vertex E, B corresponds to F, D to H, and C to G. Then, side AD corresponds to side EH, AB to EF, etc.; then $\angle A$ corresponds to $\angle E$, $\angle D$ to $\angle H$, etc.

2. Have pupils observe that the figures have the same shape, but differ in size.

3. Elicit that corresponding angles are equal.

4. Elicit that although corresponding sides are not equal, the ratios of the corresponding sides are equal.

   a. $\frac{\text{Length of side DC}}{\text{Length of side HG}} = \frac{5}{10}, \text{ or } \frac{1}{2}$

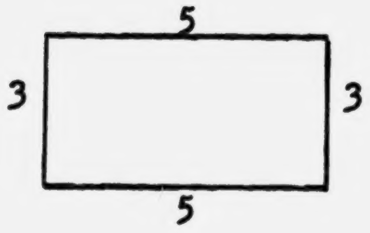
   b. $\frac{\text{Length of side AD}}{\text{Length of side EH}} = \frac{4}{8}, \text{ or } \frac{1}{2}$, etc.

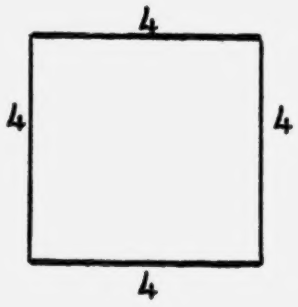
   Note: When angles and sides are said to be equal, we mean their measures are equal.

5. Have pupils answer such questions as the following:

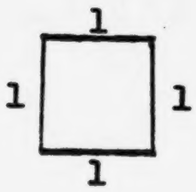
   a. These two figures have equal corresponding angles. Do you think they have the same shape?

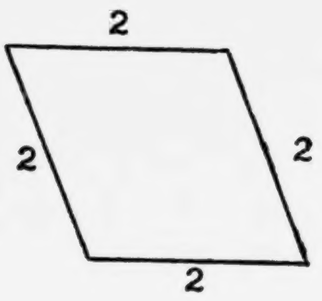
   Have pupils see that no matter what correspondence is set up between the vertices of the figures, corresponding sides will never be in proportion.





   b. These two figures have their sides in proportion. Do you think they have the same shape? Explain.





Elicit that two figures have the same shape only when, for some correspondence, corresponding angles are equal and corresponding sides are in proportion.

6. Tell the pupils that two geometric figures (polygons) are said to be similar if for some correspondence of the vertices, they

   a. have their corresponding angles equal and
   b. their corresponding sides are in proportion.

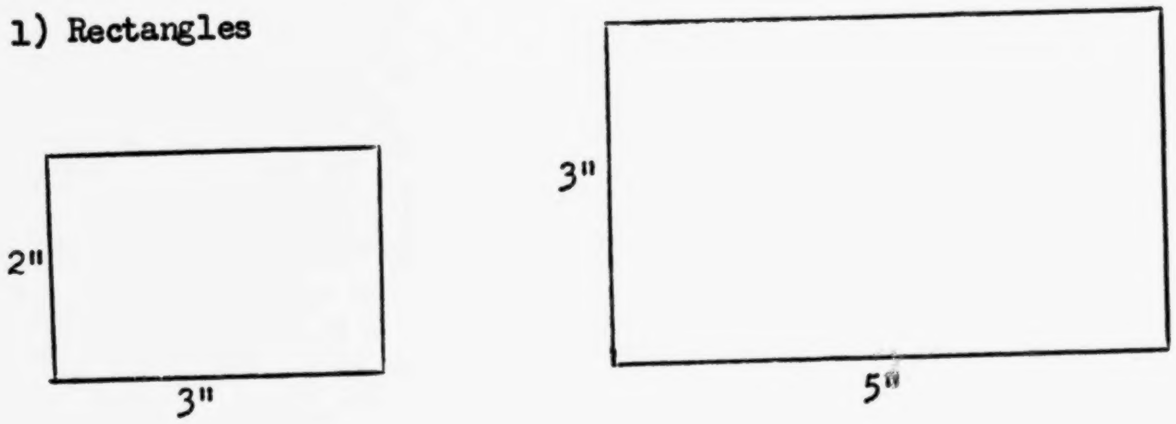
7. Have pupils practice recognizing similar figures.

   a. Are all rectangles similar? Explain.

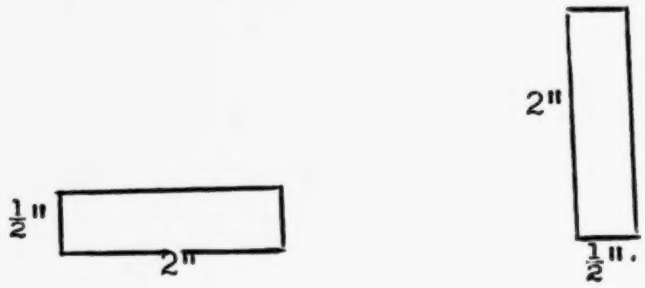
   b. Are all squares similar? Explain.

   c. Decide which of the following pairs of figures are similar. Give a reason for each answer.

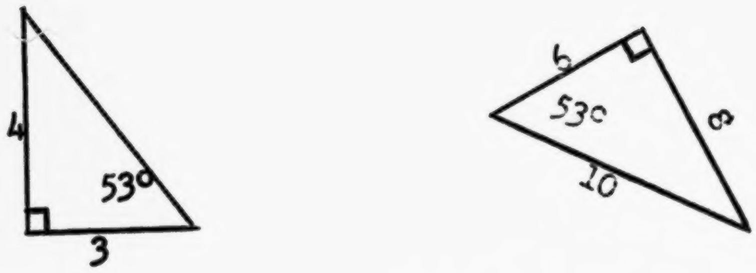
      1) Rectangles



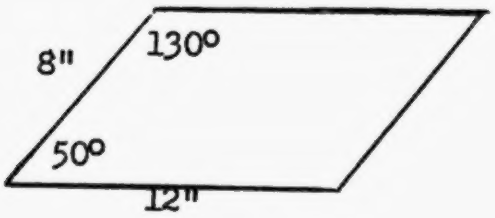
      2) Rectangles

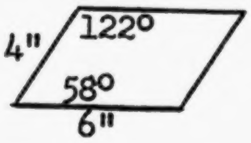


      3) Right triangles



4) Parallelograms





8. Have pupils learn that two triangles are similar when two angles of one triangle are equal to two angles of the other triangle.

   a. Have pupils use graph paper, protractors and rulers to draw the following pairs of triangles:

   Triangle ABC   AB = 2"   ∠A = 90°   ∠B = 60°

   Triangle DEF   DE = 4"   ∠D = 90°   ∠E = 60°

   Note: The pupils will discover that as they draw these triangles, the three facts given for each triangle will determine the triangle. This means that if each pupil uses the same three facts, in any order, he will get a triangle exactly the same size and shape as the triangle obtained by every other pupil.

   b. Set up a chart such as the following and have the pupils fill in the blanks, by measuring the three parts which were not given.

Triangle ABC	Triangle DEF
∠A = 90°	∠D = 90°
∠B = 60°	∠E = 60°
∠C =	∠F =
AB = 2"	DE = 4"
AC =	DF =
CB =	FE =

c. Elicit that these two triangles are similar because when we let vertex A correspond to vertex D, vertex B to vertex E, and vertex C to vertex F, then

1) corresponding angles are equal
2) corresponding pairs of sides have a constant ratio (are in proportion).

9. Have pupils construct a triangle using any length of his own choosing for a base line, but using the same angles as in 7-a.

a. Have the pupils compare the lengths of the sides of △ ABC with the lengths of this new triangle. (labeled A'B'C')

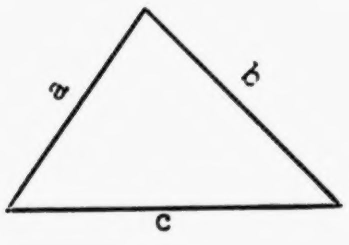
$$\frac{AB}{A'B'} \qquad \frac{AC}{A'C'} \qquad \frac{CB}{C'B'}$$

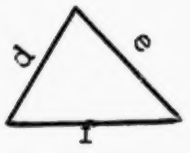
b. Have pupil conclude that the ratios are the same and therefore the sides are in proportion.

c. As a result of these experiments, have the pupils conclude that in the case of triangles, if the angles of one triangle equal the angles of another triangle, (two pairs are sufficient), the triangles are similar. It appears that if the angles are equal, the corresponding sides will automatically be in proportion.

Elicit that this does not hold for other figures, just triangles. Note rectangles in 6.

Note: Proportionality of corresponding sides of two similar triangles, such as the following, may be expressed in more than one way:





If a corresponds to d, b to e, and c to f, then

$$\frac{a}{d} = \frac{c}{f} \qquad \text{or} \qquad \frac{a}{c} = \frac{d}{f}$$

As an optional exercise, pupils may be asked to prove this.

10. Have pupils practice identifying similar triangles when the angles are given. (Give several examples.)

11. Have pupils learn to use properties of similar triangles in indirect measurement.

    a. Tell pupils story of Thales, a Greek mathematician, who lived about 600 B.C. On a trip to Egypt, he asked the height of one of the pyramids. No one knew. He soon astounded the Egyptians by computing the approximate height, using a method which involved the shadow of the pyramid.

       Today many Boy Scouts use the same method to find the height of a tree or building. This method is called "shadow reckoning."

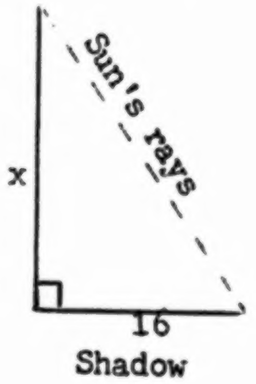
    b. Pose problem:

       A flagpole casts a shadow 16 ft. long at the same time a nearby post 5 ft. tall casts a shadow 2 ft. long. Find the height of the flagpole.

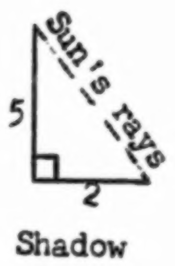
       1) Discuss the difficulty one would have in measuring the flagpole directly and the need for a method of indirect measurement.

       2) Have the pupils draw and label a diagram (this is a diagram, not a scale drawing) showing the conditions of the problem.

          Flagpole:



Post:



       3) Discuss with pupils the great distance to the sun. Because of this, the sun's rays may be considered parallel near the earth, and so the corresponding angles formed in the diagram may be considered equal. The triangles are then similar.

       4) Discuss, too, that as the sun goes down, the shadows get longer. It is for this reason that the problem must state that the time of day is the same. Why must the two objects be located close to each other?

5) Have pupils set up a proportion between the corresponding sides of the two similar triangles.

$$\frac{\text{Height of flagpole}}{\text{Height of post}} = \frac{\text{Length of shadow of flagpole}}{\text{Length of shadow of post}}$$

Have pupils make the proper substitutions: $\frac{x}{5} = \frac{16}{2}$

Multiplying both sides of the equation by 5: $x = \frac{80}{2}$

$x = 40$

The approximate height of flagpole is 40 feet.

B. Suggested Practice

1. Draw the diagram and explain the reasoning that Thales used to solve the pyramid problem, according to the following story:

   He set up a stick of known length near the pyramid. He waited until the shadow of the stick equaled its length. He then measured the length of one side of the square base of the pyramid, and the length of the shadow of the pyramid. The height of the pyramid equaled the length of the shadow, plus one-half the length of the base.

2. An apartment building casts a shadow 75 ft. long when a telephone pole 22 ft. high casts a shadow 10 ft. long. How high is the apartment building?

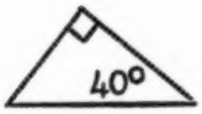
3. The shadow of a church steeple on level ground is 25 ft. long. At the same time, an 8 ft. lamp post casts a 5 ft. shadow. What is the height of the steeple?

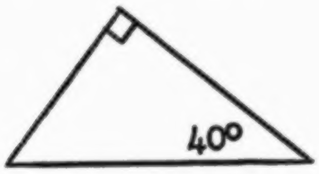
4. A tree casts a shadow 12 ft. long. At the same time, a man 6 ft. tall casts a shadow of 3 ft. What is the height of the tree?

5. The sides of a triangle are 3, 10 and 12 inches respectively. The shortest side of a similar triangle is 12 inches. Find the length of the other sides of the second triangle.

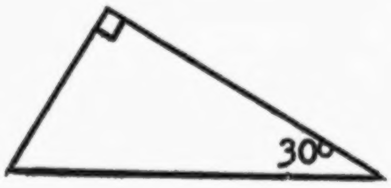
6. If one acute angle of a right triangle is equal to one acute angle of another right triangle, the triangles are similar, since they already agree in their right angles. Tell which of these pairs of triangles are similar. Explain in each case.

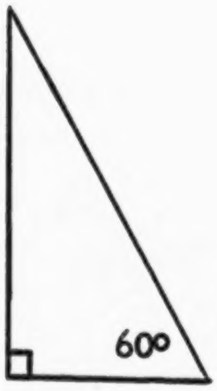
   a.





b.





V. Numerical Trigonometry of the Right Triangle (Tangent Ratio)

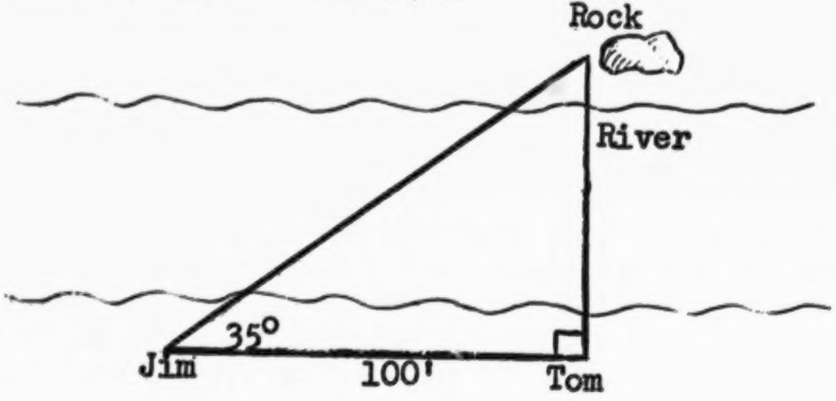
A. Suggested Procedure

1. Have pupils recall that they learned how to solve problems involving indirect measurement through the use of formulas, scale drawings, and similar figures.

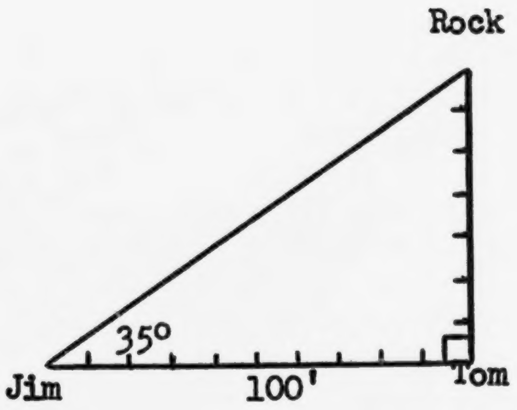
   a. Pose problem such as:

      1) Tom and Jim, standing on one side of a river, decided they wanted to measure the distance across the river. Tom placed himself directly opposite a large rock on the other side. Jim walked along the river bank (straight line) until he reached a point 100 feet from Tom. Jim measured the angle noted as J in the diagram. He found it to be 35°. What is the distance across the river? (Angles may be measured by protractor mounted on a board, transit, etc.)

      Note: Jim walked in a line which is at right angles to the line between Tom and the rock.



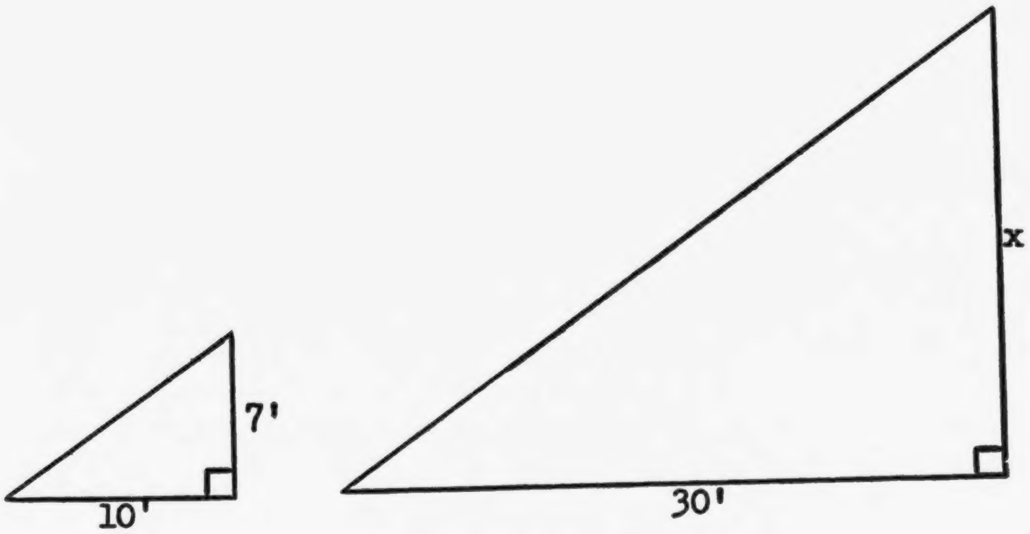
Help pupils solve the problem using scale drawing.



Since each unit on the scale is equal to 10', the pupil finds that the distance across the river is about 70'.

2) At 4:00 p.m., a pole 7 feet high casts a shadow of 10 feet. At the same time, a tree nearby casts a shadow of 30 feet. How high is the tree?

Solution: Help pupils solve this problem through the use of similar triangles.



$$\frac{7}{x} = \frac{10}{30}$$

$$x = 21$$

or

$$\frac{7}{10} = \frac{x}{30}$$

$$10x = 210$$

$$x = 21$$

The tree is 21 feet high.

b. Have pupils discuss the method used in solving the problems. He observes that two methods of indirect measurement were utilized: scale drawing, similar triangles.

c. Have pupils realize that from some known or direct measurements they were able to find other measurements indirectly.

2. Have pupils realize that similar triangles are the basis for the trigonometry of the right triangle.

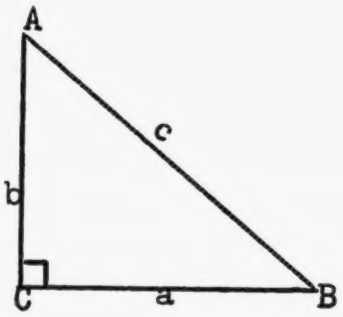
   a. Help pupils identify the parts of a right triangle, ABC. Have them recall that the legs of a right triangle are the sides that form the right angle. The third side is called the hypotenuse.

   Angle C is the right angle

   c designates the length of the hypotenuse

   b designates the length of the side opposite ∠B (also the length of the leg adjacent to ∠A)

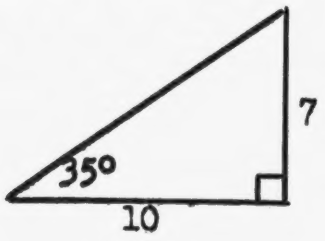
   a designates the length of the side opposite ∠A (also the length of the leg adjacent to ∠B)



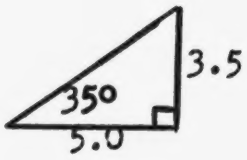
   Have pupil practice selecting the parts of various right triangles. (Place the right angle in different positions and use a variety of letters to name the vertices.)

   b. Have pupils note that in the problem 1-a, the ratio of the side opposite the 35° angle to the side adjacent to the 35° angle was $\frac{70}{100}$ or .70.

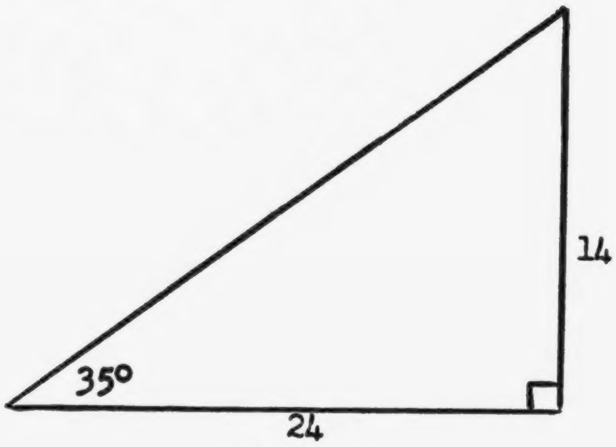
   c. Have pupils draw other right triangles on graph paper, each with an acute angle of 35°. Help them see that the ratio of the length of the leg opposite the 35° angle to the length of the leg adjacent to it is always about .70.



$\frac{7}{10}$ = .7 or .70



$\frac{3.5}{5.0}$ = .70



$\frac{14}{20}$ = .70

Have pupils generalize that for a 35° angle in a right triangle, this ratio is always the same. (Approximately .70)

Have pupils justify the conclusion that the values of the ratios are the same in each case, through his knowledge of the properties of similar triangles (corresponding sides are in proportion).

Have them realize that knowing this ratio enables them to determine an unknown distance very easily, provided we can obtain a right triangle with one leg as the unknown distance, the other leg as a known distance, and an acute angle of 35°.

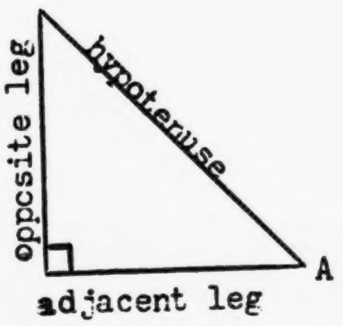
However, it may not always be convenient to obtain a 35° angle. It is, therefore, desirable to determine the ratio of the leg opposite to the leg adjacent for other acute angles.

d. Have pupils find the ratio of the length of the leg opposite to the length of the leg adjacent for angles of 31°, 45°, 72°, etc. in right triangles as in 2-c.

e. Have them realize that in each right triangle the ratio of the length of the leg opposite to the length of the leg adjacent is constant for any particular acute angle. This is true because every right triangle containing a certain acute angle is similar to every other right triangle containing an angle of the same size.

f. Tell pupils this ratio is called the tangent ratio. It may be approximated as a fraction or as a decimal. For a particular angle A, we designate this ratio as tan A (tangent of angle A). Thus,

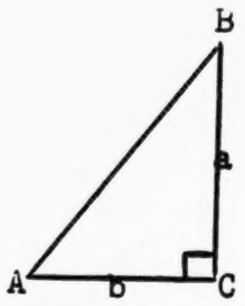
$$\tan A = \frac{\text{length of leg opposite } \angle A}{\text{length of leg adjacent } \angle A}$$



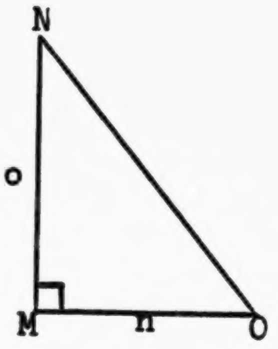
The pupils should be made aware that the above definition applies to an acute angle A of a right triangle. Thus, $0^\circ < A < 90^\circ$. There is a one-to-one correspondence between A and tan A.

g. Reinforce the concept of the tangent ratio by having the pupils do exercises such as the following:

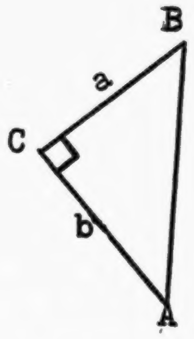
1)



tan A = ?
tan B = ?

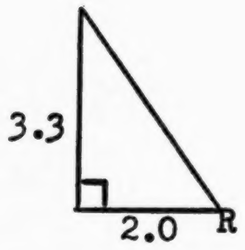


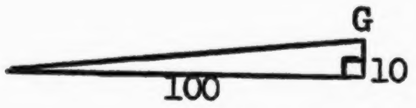
tan N = ?
tan O = ?

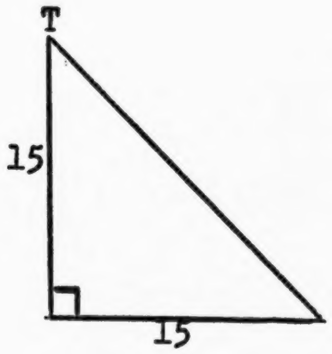


tan B = ?
tan A = ?

2) What is the tangent ratio of each of the labeled angles?







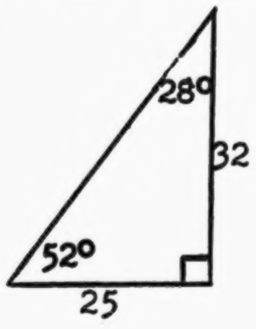
3) Write the tangent ratios for each acute angle in these triangles. Express as a 3-place decimal.

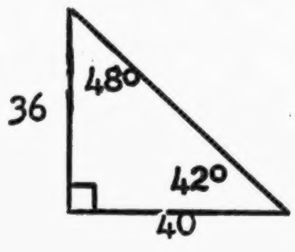
tan 28° =

tan 52° =

tan 48° =

tan 42° =

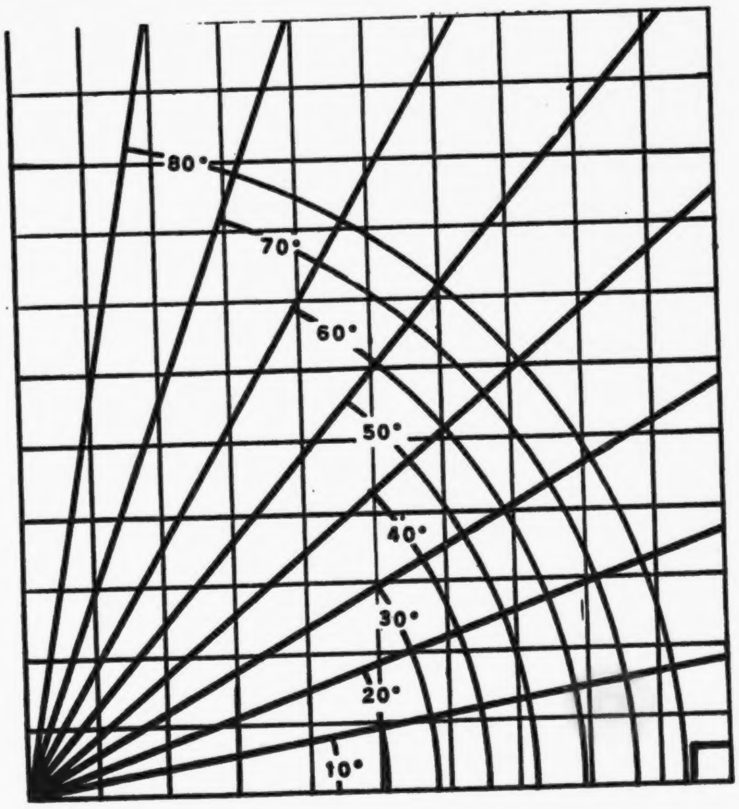




h. Help pupils discover that the value of the tangent ratio increases as the size of the angle increases by having him develop a table of values of the tangent ratio.

Have pupils use a base line 10 units long on squared paper.

Have them use a protractor to draw angles as follows:



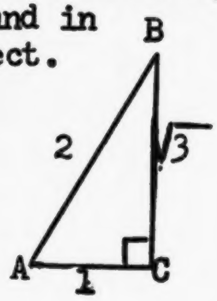
Have them see that as the angle increases in size, the value of the tangent ratio also increases.

Angle A	Length of Leg Opposite ∠A	Length of Leg Adjacent to ∠A	$\frac{\text{Length of Leg Opposite}}{\text{Length of Leg Adjacent}}$
10°	1.8	10	.18
20°	3.6	10	.36
30°	5.7	10	.57
40°	8.4	10	.84
etc.			

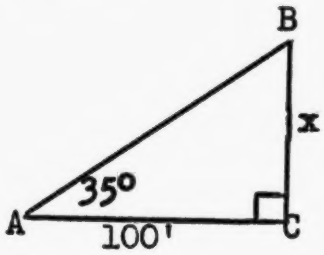
i. Help pupils read and interpret a table of tangent values. Have them compare the values in the table with the ratios found by using graph paper.

Note: Since most tangent ratios are irrational numbers, the decimal representations of these ratios found in printed tables are only approximately correct.

For example, the tangent of angle A in the figure at the right is exactly $\frac{\sqrt{3}}{1}$ or $\sqrt{3}$. A rational approximation of the tangent of angle A is 1.7321.



j. Help pupils apply tangent ratio to the solution of the problem posed in 1-a-1).



Solution:

Have pupils draw a diagram for the problem.

Have pupils discuss which sides of the triangle are involved in the problem. They then apply the tangent ratio.

$$\tan A = \frac{\text{length of leg opposite } \angle A}{\text{length of leg adjacent } \angle A}$$

$$\tan 35^\circ = \frac{x}{100}$$

$$.700 = \frac{x}{100}$$

$$100(.700) = x$$

$$70 = x$$

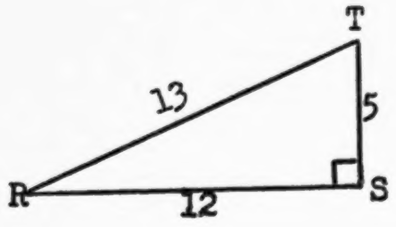
k. Have pupils practice solving problems using the tangent ratio.

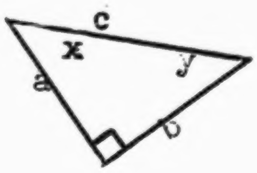
l. Have pupils become familiar with the meaning of angle of elevation and angle of depression. (Use transit.)

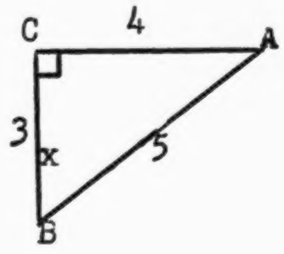
B. Suggested Practice

1. Have pupils practice finding tangent ratios, given various angles; finding angles, given tangent ratios. (Use tables.)

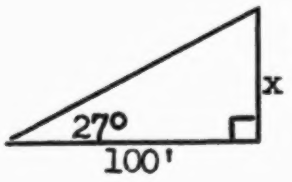
2. Have pupils practice selecting the tangent ratio for each of the acute angles in the given right triangles.

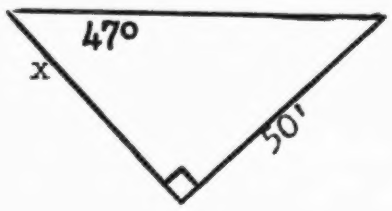


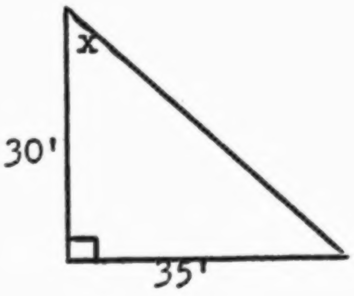




3. Have pupils use the tangent ratio to find x in each of the following diagrams:







4. Have pupils draw a diagram and use the tangent ratio in the solution of each of the following problems:

   a. To find the height (AC) of the Empire State Building, a point (B) was located 920 feet from the foot (C) of the building. ∠CBA was found to be 58°. Using these data, find the height of the building.

   b. Find the height of a tree which casts a shadow 12 feet long when the sun's rays make an angle of 70° with the ground.

   c. A tree casts a forty-foot shadow at the time when the sun's rays make an angle of 37° with the ground. What is the height of the tree?

   d. A flagpole casts a fifty-foot shadow when the sun's rays strike the ground at an angle of 40°. What is the height of the flagpole?

   e. Select additional problems from suitable textbooks.

VI. Development of Sine Ratio

A. Suggested Procedure

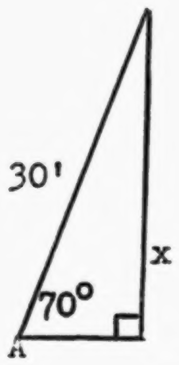
1. Have pupils see the need for a ratio other than the tangent ratio.

   a. Pose a problem such as:

      A ladder can be used with greatest safety when it is placed so that it forms an angle of approximately 70° with the ground.

      Find the height above the ground that a 30 foot ladder can reach if placed in this position.

      Solution:



      Have pupils draw a diagram of the problem.

   b. Have pupils see that the length of a leg of a right triangle is to be computed, given the hypotenuse and an acute angle.

      Have them realize that the tangent ratio is not helpful in a situation such as this, for the tangent does not use the hypotenuse.

2. Help pupils develop the ratio of the length of the leg opposite an acute angle to the length of the hypotenuse of a right triangle.

   Have them use the same procedures as those employed in the development of the tangent ratio.

3. Have pupils use the word sine (abbreviated sin) to describe the ratio developed. For an acute angle A of a right triangle

$$\sin A = \frac{\text{length of leg opposite } \angle A}{\text{length of hypotenuse}}$$

This ratio remains constant.

4. Help pupils read and interpret a table of sines.

5. Have pupils now solve problem posed in 1 by the use of the sine ratio.

6. Have them practice solving problems involving the sine ratio.

B. Suggested Practice

1. Find each of the following:

   $\sin 20^\circ$
   $\sin 30^\circ$
   $\sin 53^\circ$
   $\sin 18^\circ$
   $\sin 4^\circ$
   $\sin 80^\circ$

2. Find from the table the angles whose sines are:

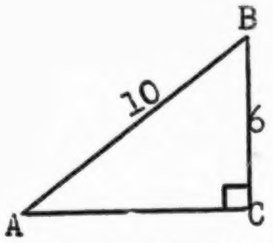
.407 .052 .970

3. As an angle increases in size, what happens to the value of the sine?

4. Complete the following by referring to the right triangle ABC:

$\sin A = \frac{6}{?} = .600 \quad A = ?^\circ$

$\sin B = \frac{?}{10} = \quad ? \quad B = ?^\circ$



5. How high is David's kite if the 200 foot string to which it is attached makes an angle of 52° with the ground? (Assume that the string is straight.)

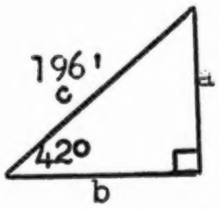
6. Jack went 300 feet into a tunnel which slopes downward at an angle of 6 degrees. How far beneath the surface was he?

7. Arthur's kite string is 196 ft. long and makes an angle of 42° with the ground. How high is the kite above the ground? Which of the following would you use to solve the problem? Why?

$\tan 42^\circ = \frac{a}{b}$

or

$\sin 42^\circ = \frac{a}{c}$



8. Select additional problems from suitable textbooks.

VII. Development of Cosine Ratio

A. Suggested Procedure

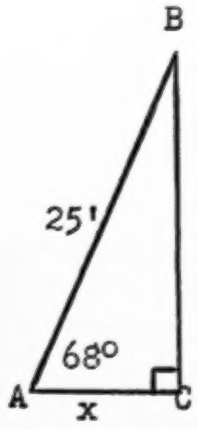
1. Help pupils see the need for the cosine ratio.

a. Pose a problem such as:

A 25-foot guy wire attached to a pole in a circus tent makes an angle of 68° with the ground. How far from the foot of the pole does the wire meet the ground?

Solution:

Have pupils draw a diagram of the problem.



b. Have pupils realize that the measurements given do not involve the direct use of either the tangent ratio or the sine ratio. Another ratio involving the leg adjacent to the acute angle and the hypotenuse may be used to solve the problem.

2. Help pupils develop this ratio by using procedures similar to those employed in the development of the tangent ratio.

3. Have them use the word cosine (abbreviated cos) to describe this ratio. For an acute angle A of a right triangle

$$\cos A = \frac{\text{length of leg adjacent to } \angle A}{\text{length of hypotenuse}}$$ This ratio remains constant.

4. Have them now solve the problem in 1, using the cosine ratio.

5. Help pupils see the relationship between cosine A and sine B in the problem posed in 1.

   Angles A and B are complementary angles (their sum is 90°).
   Then, cos A = sin B, i.e., cosine A is the sine of the complement of A.

   Since $\angle A = 68°$, then $\angle B = 22°$. The pupil may then use the sine ratio to solve the problem.

6. Help pupils read and interpret a table of cosine values.

7. Have them practice solving problems involving the cosine ratio.

B. Suggested Practice

1. Find each of the following from the table:

   cos 39° cos 14° cos 60° cos 89°

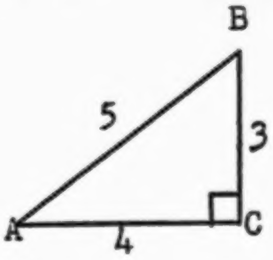
2. Find the angles whose cosines are:

   .766 .961 .500

3. Complete the following by referring to the right triangle ABC:

$\cos A = \frac{4}{?} = ?$ $\angle A = ?^\circ$

$\cos B = \frac{?}{5} = ?$ $\angle B = ?^\circ$



4. A 20-foot ladder reaches the wall at a point 18 feet above the ground. What angle does the ladder make with the house?

5. A diagonal path across a rectangular lot is 180 ft. long. The longer side of the lot is 140 ft. long. Find the angle this longer side makes with the diagonal path.

6. Use the table to find the following:

$\cos 8^\circ =$
$\cos 69^\circ =$
$\cos 80^\circ =$
$\cos 89^\circ =$

As the size of the angle increases, what happens to the value of the cosine?

7. Find, to the nearest degree, the angle for which

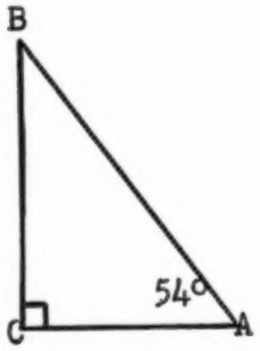
$\cos A = .857$
$\cos B = .974$
$\cos A = .242$
$\cos A = .017$

8. Refer to the diagram at the right:

a. AC = ? (to the nearest foot)

b. BC = ? (to the nearest foot)

Check using the Law of Pythagoras.



9. For right triangle ABC

write the cosine ratio for acute angle A
write the sine ratio for acute angle B

What statement would you make concerning cos A and sin B? Would this be true of cos B and sin A? Explain.

10. Is the tangent of 50° angle twice as great as the tangent of a 25° angle? Explain.

11. State whether each of the following is true or false:

    a. If sin A = .988, A is closer to 90° than to 45°.

    b. The tangent of 45° is 1.

    c. The sine of an angle of 1° is 1.00.

    d. As an angle increases in size, both the sine and cosine increase.

    e. The sine of an acute angle can be as large as 1.00.

12. A railroad track slopes at an angle of 8° to the horizontal. What vertical distance does it rise in a horizontal distance of 1 mile (5280 feet)?

13. A rectangle is 161 feet wide and 400 feet long. What angle does the diagonal make with each side of the rectangle? What is a good way to check your answers?

14. In the rectangle ABCD, the angle between AD and AC is 31° and AD = 16.3 inches. How long is AC?

15. John's kite string is 240 feet long and makes an angle of 48° with the horizontal. How high is the kite? (Assume the string is straight.)

16. Make up 3 problems in such a way that you would use the tangent ratio to solve the first, the sine ratio to solve the second, and the cosine ratio to solve the third.

17. A television tower in the town is 100 feet high and the observer is 86.9 feet from the base of the tower. What is the angle of elevation to the top of the tower for the observer?

18. From the top of a cliff 3500 feet above a lake, the angle of depression of the nearest shore is 18°. Find the distance through the air from the top of the cliff to the edge of the lake.

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